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XCVIII. *Notes on Surface-Tension.* By ALFRED W. PORTER, D.Sc., F.R.S., F.Inst.P., Emeritus Professor of Physics in the University of London*.

V. *On Jaeger's Maximum Pressure Method.*

IN determinations of the surface-tension of mercury by means of Jaeger's method (*i.e.*, the determination of maximum pressure in a gas-bubble before bursting occurs) an ambiguity arises. Sometimes the collapse of the bubble seems to be determined by the inside diameter of the delivery tube, sometimes by the outside diameter. Owing to the impossibility of seeing the growing bubble through the mercury, when this is the liquid, the choice of diameter to be used in the necessary calculation has to be determined by the experimental value of surface-tension expected. The following considerations appear to provide a criterion which removes this undesirable indeterminateness.

At the same time the usual assumption, that the results are always independent of the angle of contact, is shown to be erroneous. How this belief ever obtained credence is a matter of some surprise; it seems, however, to be very commonly held.

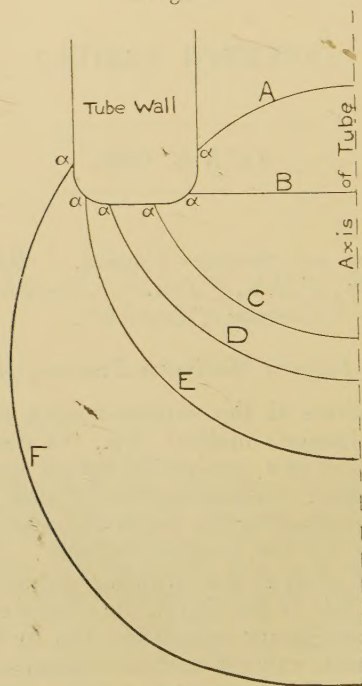
The first cases that will be considered are those in which the angle of contact (reckoned in the liquid) lies between

* Communicated by the Author. For previous parts see *Phil. Mag.*, March and August 1929.

90° and 180° , as is the case for mercury, for which it is in the neighbourhood of 135° .

As the mercury is forced down the capillary tube, and in it a bubble of gas is ultimately formed, any *equilibrium* position of the bubble must be consistent with the angle of contact between the liquid and glass being of this constant value. Taking into consideration that even the sharpest ground surfaces are terminated by rounded edges, various possible

Fig. 1.



positions of the bubble are as shown in fig. 1. With a small pressure the liquid surface is inside the tube and is convex upwards (A); with increased pressure it reaches position B, making the contact angle α with the rounded edge, and is then absolutely flat; the pressure inside must now be the same as the pressure at the same level outside the tube. With further increase of pressure the bubble begins to form with concave side upwards, and the point of contact moves along the rounded edge until the position C is reached. It will be assumed that the rounded edge is so

sharp that the radius of the line of contact in the C position can be taken as the internal radius of the tube. It now becomes a question as to whether it will require a greater pressure to cause the interface between liquid and gas to expand into positions such as D, E, F while the angle of contact remains constant. If the equilibrium pressures at D, E, F are less than at C, the bubble becomes unstable with the pressure that has increased it to C, and it will expand indefinitely. Thus the maximum pressure in this case is the pressure corresponding to the interface C, and the surface-tension can be calculated from the form of that interface. It is clear that only the internal radius will enter into the calculation. It turns out that the equilibrium pressure for the position D (just inside the curved *outer* edge) is always less than that at C (at any rate for such radii as can usefully be employed—the statement may not be true for very large radii), and consequently the expansion from C to D is always of a non-equilibrium type. At E, however, the equilibrium value may be less or greater than at C; in the latter case the expansion is resisted, and a fresh increase of pressure is required for further expansion to occur. It must be explained that the interface E is the one which has a vertical tangent at the line of contact; the line of contact is therefore such a position on the rounded outer edge that a vertical tangent corresponds to the true angle of contact; the radius of the line of contact is therefore sensibly the same as the external radius of the tube. Between E and F the equilibrium pressure is found always to diminish (except possibly for very large tubes). Hence the maximum pressure made use of will be either the pressure for case C or for case E, according to which pressure is the larger of the two.

The radius of the line of contact in any position will be denoted by C , the angle of contact by α , the maximum semi-diameter of bubble in any position by X ; the coordinates of any point by y and x . It will be assumed that a sufficiently accurate expression for the inquiry is the Poisson-Rayleigh formula

$$y = X - \sqrt{X^2 - x^2} + \frac{1}{3}X^3 \log \frac{X + \sqrt{X^2 - x^2}}{2X},$$

all distances being given their reduced values, i. e., each true distance expressed in centimetres is to be divided by β (i. e., $\sqrt{\sigma/[g(\rho - \rho_0)]}$) before being inserted in the equation; X is (in reduced units) the maximum semi-width of the meniscus if it were extended in accordance with the

formula until it became vertical. In the same way the pressures will be expressed in terms of the reduced height of a column of the liquid concerned equivalent to it. The results obtained will then be applicable to any values of σ , g , ρ ; at any stage of the operations the equations may be transformed to ordinary c.g.s. units by making them homogeneous in regard to lengths, by multiplying the several terms by the appropriate powers of β which has the dimensions of a length.

Taking the Poisson-Rayleigh equation, we have

$$\tan \theta = \frac{dy}{dx} = \frac{x}{\sqrt{X^2 - x^2}} \left\{ 1 - \frac{X^3}{3(X + \sqrt{X^2 - x^2})} \right\},$$

and at the lowest point the meridional curvature is

$$\left(\frac{d^2 y}{dx^2} \right)_0 = \frac{1}{X} \left(1 - \frac{X^2}{6} \right) = \frac{h}{2},$$

where h is the reduced-pressure difference (inside minus outside) at the vertex. If d is the reduced depth of the tip below the surface of the liquid, we have, if $p + A$ is the pressure inside,

$$p + A = A + d + y_c + h,$$

where A = atmospheric pressure and y_c the ordinate at line of contact.

Since we may regard d as fixed, the problem consists in calculating

$$p - d = y_c + h.$$

It must be noticed that in positions A, C, and D $\theta_c = \alpha - \frac{\pi}{2}$

at contact; in position E, $\theta_c = \frac{\pi}{2}$; at F, $\theta_c = \alpha - \frac{\pi}{2}$; while at B it is zero.

Position E.

For this position and for any contact angle

$$y_c = c - \frac{1}{3}c^3 \log 2,$$

where c = ext. radius; and from the curvature at the vertex we have also

$$\frac{1}{c} \left(1 - \frac{c^2}{6} \right) = \frac{h}{2},$$

so that

$$p - d = \frac{2}{3}c + \frac{2}{c} - .231c^3$$

Hence the following table of reduced values:—

<i>c.</i>	<i>p-d.</i>		<i>c.</i>	<i>p-d.</i>
·1	20·07		·5	4·276
·2	10·13		·6	3·684
·3	6·86		·7	3·245
·4	5·25 ₃		·8	2·916

Positions between C and D.

In these cases it is necessary to know the angle of contact α .

$$\tan \theta = -\tan \alpha = \frac{c}{\sqrt{X^2 - c^2}} \left\{ 1 - \frac{1}{3} \frac{X^3}{(X + \sqrt{X^2 - c^2})} \right\};$$

whence, approximately,

$$X = \frac{c}{\sin \alpha} \left\{ 1 - \frac{c^2}{3 \tan^2 \alpha \cdot (1 - \cos \alpha)} \right\}.$$

From X can be calculated for any value of c

$$h = \frac{2}{X} \left\{ 1 - \frac{X^2}{6} \right\},$$

and

$$y_c = X - \sqrt{X^2 - c^2} + \frac{1}{3} X^3 \log \frac{X + \sqrt{X^2 - c^2}}{2X};$$

and thence $p-d$. This has been done for $\alpha = 120^\circ, 135^\circ, 150^\circ$.

$$\alpha = 120^\circ.$$

<i>c.</i>	<i>X.</i>	<i>p-d.</i>		<i>c.</i>	<i>X.</i>	<i>p-d.</i>
·1	·115	17·46		·4	·454	4·47
·2	·230	8·72		·6	·672	3·01
·3	·342	5·91		·8	·876	2·43

$$\alpha = 135^\circ.$$

<i>c.</i>	<i>X.</i>	<i>p-d.</i>		<i>c.</i>	<i>X.</i>	<i>p-d.</i>
·1	·141	14·14		·5	·672	2·89
·2	·280	7·11		·6	·789	2·52
·3	·417	4·78		·7	·895	2·22
·4	·548	3·63		·8	·990	1·965

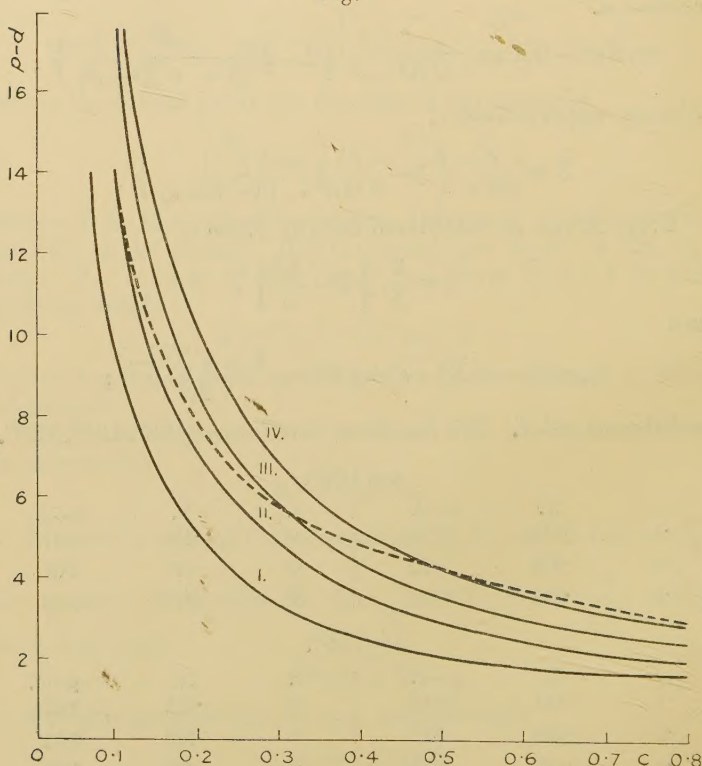
$$\alpha = 150^\circ.$$

<i>c.</i>	<i>X.</i>	<i>p-d.</i>		<i>c.</i>	<i>X.</i>	<i>p-d.</i>
·1	·199	9·75		·5	·866	2·11
·2	·392	5·01		·6	·968	1·90
·3	·571	3·25		·7	1·032	1·72
·4	·731	2·60				

The values in these tables are plotted in fig. 2 as follows:—

Curve.	From C to D.	At E.
I.	$\alpha = 150^\circ$	$\alpha = \text{any value.}$
II.	$\alpha = 135^\circ$	
III.	$\alpha = 120^\circ$	
IV.	$\alpha = 90^\circ$	

Fig. 2.



[Note: Curve IV., which was calculated for any α at E, is identical with the curve for $\alpha = 90^\circ$ at C to D.]

The use of the curves will now be explained.

Suppose $\alpha = 135^\circ$, which is approximately the value for mercury against air. Find the p on Curve II. corresponding to $c = \text{int. rad.}$ Find the pressure on Curve IV. corresponding to $c = \text{ext. rad.}$ If the latter is less than the

former, the pressure in position C is more than sufficient to expand the bubble beyond E; but if more, then the pressure must be further increased until it has the equilibrium value at E. The equilibrium value at F is always (at least within a range which is of practical use) less than at E, so it does not alter one's conclusions. The curve F is shown as a dotted line in fig. 2. The diagram is useful to interpolate from. In practice, however, the chief use is to enable a choice of suitable radii of tip to be made before actual use, and only rough values are required, because the choice should always be made so as to remove any possible doubt arising from the approximations made in calculating the curves. A table of selected values may therefore prove more useful. The accompanying table gives the radii which correspond to a series of selected equilibrium pressures—all being still expressed in "reduced" units. Values for Curves I., II., III., IV. correspond to points between C and D for the angles of contact named, while IV. is also valid for E and any angle of contact.

Values of c .

Red. Pressure.	I. 150°.	II. 135°.	III. 120°.	IV. 90°.
14	·07	·10	·12	·14
12	·08	·11	·14	·16 ₅
10	·09 ₅	·13 ₅	·17	·20 ₅
8	·12	·17 ₅	·22	·26
6	·17	·24	·29	·34 ₅
4	·25	·36	·44	·53
3	·33	·48	·60	·75

Examples:—For mercury at moderate temperatures $B = \cdot 187$; hence an inner radius of $\cdot 032$ cm. corresponds to $c = \cdot 172$. The table shows that for $\alpha = 135^\circ$ this radius corresponds to an equilibrium pressure 8. If the outer reduced radius were $\cdot 26$, the last column shows that the pressure would still be 8. It is a moot point then as to whether the inside or outside radius would be the determining one. But if the outer radius were $\cdot 345$, the equilibrium pressure thereat would only be 6; hence the pressure 8 which had been impressed at C would be sufficient to cause indefinite expansion of the bubble. In 1926 Sauerwald and Drath (*Z. f. anorg. u. algem. Chemie*, cliv. p. 79 (1926)) obtained the surface-tension of mercury with several tips, and in every case found that the *outer* radius was the effective one,

i. e., calculations from it gave the value of the surface-tension as obtained by other workers by other methods. They give no reason, however, for this being the case. The above table shows that this must be the case for their tubes if the angle of contact is not far from 135° . The following table gives the inner (r_1) and outer (r_2) radii (reduced) for their tubes, and the corresponding reduced pressures as determined from the above data :—

r_1 .	r_2 .	Pressures.	
		Inner.	Outer.
·172	·225	8	9·1
·255	·342	5·7	6·0
·132	·166	10·4	12·0
·133	·164	10·4	12·0
·158	·204	8·9	10·0
·166	·214	8·4	9·5
·141	·196	9·6	10·3

The numbers in the last column are in every case greater than those in the last but one. It would have been safer to have chosen the radii so that the pressures would have been more decidedly different—*e. g.*, for the second row, with $r_1 = \cdot 255$, r_2 should be not more than $\cdot 32$ for safety. It may be added that if the inner and outer radii are nearly alike, and mercury is the liquid, the outer edge is generally the effective one.

To calculate the surface-tension when the external edge is the effective one we have (position E)

$$p - d = \frac{2}{3}c + \frac{2}{c} - \cdot 231c^3.$$

Let $(p-d)\beta = H$, $c\beta = r_2 = \text{external radius}$;

$$\text{then} \quad \beta^2 = \frac{Hr_2}{2} - \frac{r_2^2}{3} + \frac{\cdot 231}{2} \frac{r_2^4}{\beta^2},$$

and inserting an approximate value for β^2 in the denominator,

$$\beta^2 = \frac{Hr_2}{2} \left(1 - \frac{2}{3} \frac{r_2}{H} + \cdot 461 \frac{r_2^2}{H^2} \right),$$

and

$$\beta^2 = \sigma / (g(\rho - \rho_0)).$$

This formula differs in the last term from five formulæ that have previously been given. Whether this last term is of much significance is doubtful, although it is based on the

Poisson-Rayleigh formula, which should justify it being retained. For small values of r_2 it can obviously be neglected; for larger values it is possible that uncertainties arise in connexion with the approximations made. The precise way in which the bubble breaks away may cause a fluctuation in the value of H . In any case, the problem is ultimately a kinetic, and not a static one. The possible range in H should be determined by experiment for slowly and quickly formed bubbles respectively, in order that some estimate of the possible error due to viscosity or inertia can be made.

It will be seen that when the external radius is the determining one the calculation of the surface-tension does not involve the angle of contact. *It is quite otherwise when the maximum pressure is determined by the internal radius.* Since it is now a case for numerical calculation, we will express in terms of true lengths, and the true head corresponding to $p-d$ will be called H . Then, the formulæ for position C give

$$H = -\frac{X}{3} + X \left(1 - \sqrt{1 - \frac{r_1^2}{X^2}} \right) + \frac{2\beta^2}{X} + \frac{X^3}{3\beta^2} \log \left\{ \frac{1}{2} \left(1 + \sqrt{1 - \frac{r_1^2}{X^2}} \right) \right\},$$

whence

$$\beta^2 = \frac{HX}{2} \left\{ 1 - \frac{2X}{3H} + \frac{X}{H} \sqrt{1 - \frac{r_1^2}{X^2}} - \frac{2}{3} \frac{X^2}{H^2} \log \left\{ \frac{1}{2} \left(1 + \sqrt{1 - \frac{r_1^2}{X^2}} \right) \right\} \right\},$$

where

$$X = \frac{r_1}{\sin \alpha} \left[1 - \frac{r_1^2}{3\beta^2 \tan^2 \alpha (1 - \cos \alpha)} \right].$$

In the last formula β^2 can be put approximately $\frac{HX}{2}$,

where X is approximately $\frac{r_1}{\sin \alpha}$. *This reduction requires a knowledge of the angle of contact.*

The angles of contact considered in the above lie between 90° and 180° . The question for smaller angles requires separate discussion.

XCIX. *The Numerical Solution of Partial Differential Equations.* By GORAKH PRASAD, D.Sc., Reader in Mathematics, University of Allahabad, India*.

1. Introduction.

METHODS for solving ordinary differential equations have long been known and have now found their way into text-books†. A method of solving integral equations was given by me some time ago‡. But it appears that up till now no method for solving partial differential equations numerically has been published. The aim of the present paper is to find a method for such equations.

The method explained below is very accurate, fairly rapid, and pretty general.

2. Outline of the Method.

Consider first the partial differential equation

$$\frac{\partial^2 z}{\partial x \partial y} = f(x, y, z), \quad . \quad . \quad . \quad . \quad (1)$$

and suppose that we want that solution of this equation, which is equal to $\phi(y)$ when $x=a$, and which is equal to $\psi(x)$ when $y=b$, where ϕ and ψ are given functions. Of course, if $z_{0,0}$ is the value of z when $x=a$ and $y=b$, we must have

$$\phi(b) = \psi(a) = z_{0,0}.$$

Suppose that we want the solution of (1) in the region $a < x < A$, $b < y < B$, it being assumed that a solution exists whose various differential coefficients up to the 4th, or the 5th, order are continuous. The solution will be obtained step by step. Suppose that a solution is in progress, and we have obtained the values of z and $\frac{\partial z}{\partial x}$ for a network of points given by

$$x = a + kw, \quad k = 0, 1, 2, \dots n, \quad \text{where } a + nw \text{ is nearly equal to } A;$$

$$y = b + k'w', \quad k' = 0, 1, 2, \dots r', \quad b + r'w' < B;$$

* Communicated by the Author.

† Whittaker, 'Calculus of Observations,' chap. xiv. Sanden, 'Practical Mathematical Analysis' (translated by Levy), chaps. x. and xi.

‡ See Proc. Edin. Math. Soc. xlii. pp. 46-59, where further references on the subject will be found.

and also for the points

$$x = a + kw, \quad k = 0, 1, 2, \dots, r, \quad a + rw < A, \\ y = b + (r' + 1)w'.$$

It is required to find the values of z and $\frac{\partial z}{\partial x}$ for

$$x = a + (r + 1)w, \quad y = b + (r' + 1)w',$$

which, for the sake of brevity, will be denoted by $z_{r+1, r'+1}$ and $p_{r+1, r'+1}$, respectively. These values will be expressed in terms of the values of p and z already determined and their differences. It is open for us to use either a formula involving central differences or one involving the differences lying along a sloping line in the difference table. The advantage of a central-difference formula lies in the fact that the series occurring in it is much more convergent. Indeed, it has been shown by Pearson* that the central-difference interpolation formula, from which equation (2) below is derived by integration, is so remarkably convergent that a table containing only sixty-seven logarithms, say those of the whole numbers from 34 to 100, to eight figures, is an adequate frame for finding all logarithms to seven figures of all seven-figure numbers, and for this only the first central difference, δ^2 , has to be taken into account. But the disadvantage of the central-difference formula is that, as values of $z_{k, k'}$ and $p_{k, k'}$ for $k > r$ and $k' > r' + 1$ are not known, these central differences can only be estimated; their values cannot be found exactly in the first instance. Hence a method of successive approximations has to be used; but the second approximation can always be made sufficiently accurate to enable us to dispense with higher approximations. The formula involving only the backward differences does not necessitate the use of successive approximations; but, on the whole, the central-difference formulæ are much less laborious to work with†.

The formula for calculating $p_{r+1, r'+1}$ is

$$p_{r+1, r'+1} = p_{r+1, r'} + w' \left\{ \frac{1}{2} - \frac{1}{24} \delta'^2 + \frac{11}{1440} \delta'^4 - \frac{191}{120960} \delta'^6 + \dots \right\} (s_{r+1, r'} + s_{r+1, r'+1}), \quad (2)$$

where $s_{r, r'}$ denotes the value of $\frac{\partial^2 z}{\partial x \partial y}$ when $x = a + wr$,

* K. Pearson, 'Tracts for Computers,' no. ii.

† Cf. Bosanquet, "On the Capillary Rise of Liquids in Wide Tubes," Phil. Mag. no. 28, 1928.

$y=b+w'r'$, and an approximate extrapolated value of $s_{r+1, r'+1}$ is used, as illustrated by the example below. The notation employed here for the differences is the same as that used by Pearson*.

The formula for $z_{r+1, r'+1}$ is

$$z_{r+1, r'+1} = z_{r, r'+1} + w \left\{ \frac{1}{2} - \frac{1}{24} \delta^2 + \frac{11}{1440} \delta^4 - \frac{191}{120960} \delta^6 + \dots \right\} (p_{r, r'+1} + p_{r+1, r'+1}), \quad (3)$$

where the approximate value of p found by (2) is used.

Small errors in the original extrapolated value of s will produce a much smaller error in the value computed for p , because of the factor $\frac{1}{2}w'$; and this will give rise to a still smaller error in z for a similar reason. The value z_1 thus obtained for z is, therefore, very nearly correct. The original extrapolated value of s should now be replaced by the value obtained by substituting in (1) z_1 for z and the proper values of x and y , and the computation repeated. The value of z should change only by a very small amount now. In fact, this change should be so small that if the computation were to be repeated with this changed value of z , there would be no further alteration in the value of z . We thus get the correct value of z .

We see now that the process of solving partial differential equations is analogous to that of solving a pair of simultaneous ordinary differential equations of the first order.

If very large intervals are taken, it is possible that the extrapolated values of δ^2 and δ'^2 might be appreciably wrong. In such cases the computation should be revised at every few steps, using the new (computed) values of these differences.

3. An Example.

Consider the differential equation

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{(\log_e 10)^2 \cdot z}{6x^{\frac{1}{2}}y^{\frac{1}{2}}}, \quad \dots \dots \dots (4)$$

and suppose we want that solution which is equal to $25.9546.10^{y^{\frac{1}{2}}}$ when $x=2$ and is equal to $27.6853.10^{x^{\frac{1}{2}}}$ when $y=3$. Suppose that the values of z for

$$x = 2.0, 2.5, \dots 5.0, \dots,$$

$$y = 3.0, 3.8, \dots 6.2,$$

* K. Pearson, 'Tracts for Computers,' no. iii. p. 10.

have been calculated, and that the values of z for

$$x = 2.0, 2.5, \dots 4.5,$$

$$y = 7.0$$

have also been calculated, and we want the value of z when $x=5.0$, $y=7.0$. The computed values of z , p , and s are given in the annexed tables.

TABLE I.
Values of z .

$y \backslash x.$	2.0.	2.5.	3.0.	3.5.	4.0.	4.5.	5.0.
3.0	718.562	1055.33	1493.83	2056.27	2768.53	3660.75	4767.80
3.8	943.420	1385.58	1961.29	2699.73	3634.89	4806.30	6259.77
4.6	1194.86	1754.86	2484.01	3419.26	4603.65	6087.27	7928.14
5.4	1474.43	2165.46	3035.22	4219.30	5680.83	7511.58	9783.18
6.2	1783.63	2619.57	3708.01	5104.10	6872.13	9086.80	11834.76
7.0	2123.95	3119.39	4415.52	6077.99	8183.36	10820.60	?

TABLE II.
Values of p .

$y \backslash x.$	2.0.	2.5.	3.0.	3.5.	4.0.	4.5.	5.0.
3.0	584.971	768.432	992.947	1265.41	1593.70	1986.78	2454.82
3.8	768.026	1008.89	1303.67	1661.39	2092.42	2608.50	3223.00
4.6	972.720	1277.79	1651.12	2104.19	2650.08	3303.72	4081.91
5.4	1200.32	1576.76	2037.45	2596.52	3270.14	4076.72	5037.09
6.2	1452.03	1907.41	2464.70	3141.02	3955.91	4931.62	6093.40
7.0	1729.08	2271.36	2934.99	3740.35	4710.72	5872.61	?
		121.35	141.73	165.01	191.52		
			2.90	3.23			

The figures below the main entries in these tables are the values of δ^2 and δ^4 , or δ'^2 and δ'^4 , as the case may be.

Looking at the values of δ'^2 in the column $x=5.0$ in Table III., we can guess that an approximate value of the next δ'^2 is 6.40. This leads to the value

$$2 \times 1385.78 - 1256.07 + 6.40 = 1521.89$$

of s for the blank space in which there is the note of interrogation. We put down 6.60 as the value of δ^2 in this blank space. As the value 6.30 or 6.90 would produce a difference of only one unit in the last place in the values of p , we see it is not very important that our guess be quite right.

TABLE III.
Values of s .

$y \backslash x$.	2.0.	2.5.	3.0.	3.5.	4.0.	4.5.	5.0.
3.0	215.849	283.544	366.387	466.924	588.059	733.104	905.81
3.8	242.074	317.994	410.904	523.655	659.509	822.172	1015.86 6.83
4.6	269.926	354.581	458.180	583.904	735.386	916.767	1132.74 6.45 .31
5.4	299.315	393.187	508.066	647.477	815.454	1016.59	1256.07 6.38
6.2	330.224	433.789	560.530	714.338	899.662	1121.56	1385.78
7.0	362.671	476.411	615.604	784.528	988.062	1231.77	?

By formula (2) we now find the corresponding values of p to be

$$6093.41 + 0.8 \left\{ \frac{1}{2}(1385.78 + 1521.89) - \frac{1}{24}(6.40 + 6.60) \right\} \\ = 7256.04.$$

The last row of Table II. now becomes

$y \backslash x$.	2.0.	2.5.	3.0.	3.5.	4.0.	4.5.	5.0.
7.0	1729.08	2271.36	2934.99	3740.35	4710.72	5872.61	7256.04
		121.35	141.73	165.01	191.52	221.54	
			2.90	3.23	3.51		

We put down 4.00 as the value of the next δ^4 . This gives 255.56 as the value of δ^2 under $x=5.0$. Applying now formula (3), we have, finally, for the value of z ,

$$10820.60 + 0.5 \left\{ \frac{1}{2}(5872.61 + 7256.04) - \frac{1}{24}(221.54 + 255.56) \right\} \\ = 14092.82.$$

It is easy to verify that $10^{x^{\frac{1}{2}}+y^{\frac{1}{2}}}$ is the solution of the differential equation under consideration, and that the value of z found above is quite correct.

In general, the value of z found would be slightly inaccurate. With this value of z a better value of s could be calculated to replace the assumed value 1521.89. The calculation would then have to be repeated.

We see that, although our intervals are fairly large, viz. 0.5 and 0.8, we have been able to attain seven-figure accuracy at the very first approximation. Taking larger intervals and using the terms in δ^4 and δ'^4 also in the second approximation, it is evident that we could very rapidly tabulate the values of z for a large region. All intermediate values of z could be derived by the use of bi-variate central-difference interpolation formulæ*.

4. Initial Values for z .

It remains now to see how a few initial values of z and $\frac{\partial z}{\partial x}$ can be calculated to start the solution. By differentiating successively the equation (1), we find

$$\frac{\partial^3 z}{\partial x^2 \partial y}, \quad \frac{\partial^3 z}{\partial x \partial y^2}, \quad \frac{\partial^4 z}{\partial x^3 \partial y}, \quad \frac{\partial^4 z}{\partial x^2 \partial y^2}, \quad \frac{\partial^4 z}{\partial x \partial y^3}; \dots$$

Also by differentiating $\phi(y)$ or $\psi(x)$, we find the values of

$$\frac{\partial z}{\partial x}, \quad \frac{\partial^2 z}{\partial x^2}, \quad \frac{\partial^3 z}{\partial x^3}, \dots; \quad \frac{\partial z}{\partial y}, \quad \frac{\partial^2 z}{\partial y^2}, \quad \frac{\partial^3 z}{\partial y^3}, \dots$$

at the point $x=a$, $y=b$. Thus the values of all necessary coefficients in the Taylor's expansion,

$$\begin{aligned} z_{r,r'} = z_{0,0} &+ r w \left(\frac{\partial z}{\partial x} \right)_{0,0} + r' w' \left(\frac{\partial z}{\partial y} \right)_{0,0} \\ &+ \frac{1}{2} \left\{ r^2 w^2 \left(\frac{\partial^2 z}{\partial x^2} \right)_{0,0} + 2 r r' w w' \left(\frac{\partial^2 z}{\partial x \partial y} \right)_{0,0} \right. \\ &\left. + r'^2 w'^2 \left(\frac{\partial^2 z}{\partial y^2} \right)_{0,0} \right\} + \dots, \end{aligned}$$

become known; and substituting 1, 2, ... for r, r' , we can find a sufficient number of values of z . We can do the same for p .

* K. Pearson, 'Tracts for Computers,' no. iii.

Another procedure is also possible. We might take w and w' so small in the beginning that δ^2 , δ'^2 might become negligible, and we might thus be able to use formulæ (2) and (3) from the very beginning.

Thus, in the example considered above, take

$$w = w' = 0.1.$$

We know $z_{0,0}$, $z_{0,1}$, $z_{1,0}$; $p_{0,0}$, $p_{1,0}$; $s_{0,0}$, $s_{0,1}$, $s_{1,0}$. To find $z_{1,1}$ we notice that

$$s_{0,0} = 215.849, \quad s_{1,0} = 228.292,$$

$$s_{0,1} = 219.030.$$

Therefore, supposing that $\Delta' \Delta s_{0,0}$ is constant, the extrapolated value of

$$\begin{aligned} s_{1,1} &= 219.030 + 12.443 \\ &= 231.473. \end{aligned}$$

We find, therefore, that an approximate value of

$$\begin{aligned} p_{0,1} &= p_{0,0} + \frac{1}{2}w'(s_{0,0} + s_{0,1}) \\ &= 606.715. \end{aligned}$$

Similarly

$$p_{1,1} = 641.667.$$

These values give

$$z_{1,1} = 807.689$$

by (3).

To see if a second approximation is required, we substitute this in (4). The value of $s_{1,1}$ comes out to be 231.655. As there is an appreciable difference between this and the extrapolated value, we revise the calculation with the new value of s and get

$$z_{1,1} = 807.678.$$

If we compare this with the known solution, we find that it is quite correct. We see that by taking w and w' to be about 0.1, we can use formulæ (2) and (3) from the very beginning.

5. *A more general Equation.*

If instead of (1) we have the equation

$$\frac{\partial^2 z}{\partial x \partial y} = f(x, y, z, p, q), \quad . \quad . \quad . \quad (5)$$

the above method still gives the solution, but we have to calculate in this case q also at each of the points where $z_{r,r'}$ is found. This can be done from the formula:

$$q_{r+1,r'+1} = q_{r,r'+1} + w \left\{ \frac{1}{2} - \frac{1}{24} \delta^2 + \frac{11}{1440} \delta^4 - \frac{191}{120960} \delta^6 + \dots \right\} (s_{r,r'+1}, s_{r+1,r'+1}). \quad (6)$$

Thus in this case the computation of s , p , q , and z all have to go along simultaneously, instead of the computation of s , p , and z only.

6. Transformation of Equations.

By a slight modification of Laplace's method * any partial differential equation of the form

$$Rr + Ss + Tt = F(x, y, z, p, q), \quad . \quad . \quad . \quad (7)$$

where R , S , and T are functions of x and y , can, in general, be transformed into an equation of the form

$$\frac{\partial^2 z}{\partial x \partial y} = f(x, y, z, p, q).$$

We see thus that most of the partial differential equations which are linear in the second-order terms can be easily solved numerically.

Allahabad,
October 2, 1929.

C. The Scattering Powers of the Atoms in Magnesium Oxide for X-Rays and some Related Properties. By G. W. BRINDLEY, M.Sc, Assistant Lecturer in Physics, University of Leeds †.

1. Introduction.

EXPERIMENTS have been made recently by R. W. G. Wyckoff and Miss A. H. Armstrong ⁽¹⁾ on the intensity of reflexion of X-rays by powdered crystals of magnesium oxide and sodium fluoride, and from their results they have calculated the scattering powers of the atoms in these crystals. The scattering power of an atom for X-rays is

* Forsyth, 'A Treatise on Differential Equations,' p. 504 (London, 1914).

† Communicated by Prof. R. Whiddington, F.R.S.

closely connected with the number and distribution of its electrons and with its amplitude of thermal vibration. References to work on this subject are given in a recent paper by the writer⁽²⁾, in which the amplitudes of vibration at room temperature are calculated for the atoms in the crystals NaCl, KCl, NaF, and LiF. For these crystals it was possible to assume that the atoms are singly ionized, but in the case of MgO it is not certain whether the lattice points are occupied by neutral atoms or by singly or doubly ionized atoms.

The present calculations were undertaken with the object of determining, if possible, the state of ionization of the atoms in MgO, and also of estimating their amplitudes of thermal vibration, from the experimental data of Wyckoff and Miss Armstrong.

2. *The Relation between the Scattering Power of an Atom and the Number and Distribution of its Electrons.*

The most direct way, theoretically, of determining the number of electrons associated with each atom would be to use the following equation, due originally to A. H. Compton⁽³⁾, which enables the electron distribution $U(r)$ to be calculated from the scattering power F :

$$U(r) = \frac{8\pi r}{D^2} \sum_1^{\infty} n \cdot F_n \sin 2\pi nr/D. \quad . \quad . \quad (1)$$

If F_n , the scattering power in a direction θ_n , the glancing angle of incidence for the spectrum of order n from planes of spacing D , is expressed in terms of the electron as unit, then $U(r)$ is the radial electron density at distance r from the nucleus, and the total number of electrons in the atom is $\int_0^{\infty} U(r) dr$. This method, however, suffers from the disadvantage that F_n can only be measured for a limited range of angles, and for the largest angles at which F can be measured F_n is quite appreciable. In consequence the series cannot be summed completely. This difficulty is magnified because $n \cdot F_n$ occurs in the summation, and $n \cdot F_n$ is still large when F_n is small. Whether the higher-order terms in the summation are neglected, or an extrapolation of F to zero is attempted, there is usually a very considerable uncertainty in $U(r)$.

Wyckoff and Miss Armstrong recognized these difficulties; they calculated the charge distribution along a cube edge of MgO, using an expression similar to (1), with and without

an extrapolation of F to zero, and their results were markedly different in the two cases. Their final conclusion was that "in view of the inexact nature of this extrapolation too much physical significance cannot be attached to the results of these series developments" (*op. cit.* p. 439). A short account has recently appeared of an investigation by E. O. Wollen⁽⁴⁾, who has used the radial Fourier series, and he finds that "the data indicate that the number of electrons associated with magnesium and oxygen is more nearly that of the neutral atoms than that of the ions."

In view, however, of the uncertainties inherently attached to the use of equation (1), an attempt has been made to solve the problem by a different method, using the experimental data of Wyckoff and Miss Armstrong.

3. An Alternative to the Method of Fourier Analysis.

The underlying theory, which has been fully discussed elsewhere⁽⁵⁾, is briefly as follows:—

Knowing the wave function ψ for an atom with central symmetry, $U(r)$ can be calculated:

$$U(r) = 4\pi r^2 \cdot \psi \bar{\psi} \dots \dots \dots (2)$$

F_0 , the scattering power of the atom *at rest*, is then given by the equations

$$\begin{aligned} F_0 &= \int_0^\infty U(r) \cdot \frac{\sin \phi}{\phi} \cdot dr, \\ &= 4\pi \int_0^\infty \psi \bar{\psi} \cdot \left(\frac{\sin \phi}{\phi} \right) r^2 dr, \dots \dots (3) \end{aligned}$$

where

$$\phi = 4\pi r \left(\frac{\sin \theta}{\lambda} \right).$$

F_T , the scattering power of the atom when vibrating at temperature T , is connected with F_0 by the relation

$$F_T = F_0 \cdot e^{-M}, \dots \dots \dots (4)$$

where

$$M = \left(\frac{8\pi^2}{3} \right) \cdot \bar{u}^2 \cdot \left(\frac{\sin \theta}{\lambda} \right)^2 = \alpha \left(\frac{\sin \theta}{\lambda} \right)^2.$$

This expression for M was given by I. Waller⁽⁶⁾, and is true for crystals having cubic symmetry. \bar{u}^2 is the mean square of the displacement of the atom from its equilibrium position; $\sqrt{u^2}$ may be regarded as the mean amplitude of vibration at temperature T .

The calculation of F_0 and F_T from $U(r)$ is not accompanied by the same disadvantages as the reverse calculation of $U(r)$ from F , for since in equation (3) $U(r)$ and $(\sin \phi)/\phi$ both tend to zero as r becomes large, F_0 can be calculated accurately.

The equations (2), (3), and (4) may be made the basis of a method for determining the state of ionization of the atoms in a crystal, *provided the experimental data are sufficiently extensive.*

A series of investigations by R. W. James and others on the scattering powers of the atoms in crystals of rock-salt, NaCl, sylvine, KCl, and aluminium⁽⁷⁾ at different temperatures has shown that when the scattering powers F_T at room-temperature are corrected for thermal agitation and for zero-point energy, the resulting F_0 values for the atoms at rest agree very well indeed with the theoretical F_0 values obtained by use of equation (3) from the charge distributions calculated by the method of D. R. Hartree⁽⁸⁾, which is based on wave mechanical principles. Apart from the general evidence⁽⁹⁾ showing that wave functions calculated by Hartree's method are correct to a fairly close approximation, the results obtained by James and his co-workers are strong evidence for the view that scattering powers calculated from Hartree charge distributions are approximately correct when the appropriate temperature factor, e^{-M} , is applied. This principle has been discussed more fully by the writer in a recent paper⁽²⁾.

In applying the principle to MgO, we require first the radial charge distribution and hence the F_0 functions for neutral O, neutral Mg, O^{-2} , and Mg^{+2} . Secondly, we must estimate the temperature factors for the atoms in MgO, and so calculate the F_T functions by means of equation (4); this second step is, in general, the more difficult of the two. A comparison of the theoretical F_T values and the experimental values should then decide the degree of ionization of the atoms in the crystal. Whether it will be possible or not to make the final decision will depend on whether the theoretical F_0 and F_T functions are appreciably different for the ionized and un-ionized atoms in the region of $(\sin \theta)/\lambda$ where experimental measurements have been made.

A somewhat similar investigation to the present one has been made by R. W. James, the writer, and R. G. Wood for aluminium⁽⁷⁾. In that case it was not possible to determine whether the atoms in crystalline aluminium are singly, doubly, or trebly ionized, because it is only at very small scattering angles that F_0 is appreciably different for the

three ions; this arises because the M electron(s) in Al^{+2} and Al^+ are so diffuse.

The same difficulty will undoubtedly arise for Mg^{+2} , Mg^+ , and Mg owing to the diffuseness of the M electron(s) in Mg^+ and Mg; but it is not at first sight evident whether there will be the same difficulty for oxygen, for whereas neutral Mg has two electrons *more* than the neon-like ion Mg^{+2} , neutral O has two electrons *less* than O^{-2} . It was therefore thought worth while to examine the theoretical F_0 curves for neutral O and O^{-2} .

4. *The Scattering Powers of Neutral Oxygen and of O^{-2} .*

The distribution of charge in neutral oxygen has been calculated recently by Miss Black using Hartree's method, and James has calculated the F_0 function*, the atom being assumed to be spherically symmetrical.

The case of O^{-2} was more difficult. In Schrödinger's equation for the wave-function ψ of an electron in a central field, there occurs a parameter E which, considered physically, is the total negative energy of the electron in the potential field. Hartree uses a constant ϵ , given by $E = -\frac{1}{2}\epsilon$, so that for an electron in a normal atom ϵ is essentially positive. In a calculation of the charge distribution of F^- considerable difficulty was found in obtaining the wave function for the 22 electrons owing mainly to the charge on the outside of the ion being in a very weak attractive field and partly to ϵ being very small. In the case of O^{-2} the charge on the outside of the ion would be in a repulsive field, and, from an inspection of ϵ for other neon-like ions, ϵ for O^{-2} would probably be *negative*, in which case there would be no solution of the wave equation of the required type; if ϵ is just positive a solution of the required type will exist but will be difficult to calculate.

F_0 curves, however, have been calculated for the following neon-like ions from charge distributions obtained by Hartree's method, F^- , Ne, Na^+ , Al^{+3} , and Si^{+4} , and a method was therefore sought for determining the F_0 curve for O^{-2} by extrapolation. Several quite different methods of extrapolation have been used which lead to almost identical results.

One method was as follows. The wave function obtained by Hartree's method for an electron in an atom of nuclear

* I am indebted to Miss Black and Mr. James for these data.

charge Ze can be represented approximately by considering the electron to be in a central Coulomb field due to a charge $Z'e$. Now by choosing Z' rightly, the F_0 curve calculated from the Hartree distribution of charge and the F_0' curve* obtained from the "hydrogen-like" distribution can be made almost identical. Actually it was found best to choose Z' so that for each electron in the above series of neon-like ions the Hartree and the hydrogen-like F_0 curves coincided at $F_0=0.5$. For a particular group of electrons, Z' was found to vary linearly with Z ; hence Z' could be obtained with a fair degree of certainty for O^{-2} by extrapolation. The differences between the Hartree F_0 curves and the hydrogen-like F_0' curves were small, and so by extrapolation the difference between the Hartree F_0 curve and the hydrogen-like F_0' curve for O^{-2} could be calculated. In this way the Hartree F_0 curve was estimated for each group of electrons in O^{-2} . It was found that the F_0 curve for the two $1s$ and the two $2s$ electrons was practically the same for neutral oxygen and for O^{-2} , but for a $2p$ electron in O^{-2} the F_0 curve was found to fall to zero more quickly than for a $2p$ electron in neutral oxygen. As this point is of some importance, a few details may be given of the calculation.

For a $2p$ electron in a Coulomb field of charge $Z'e$ ⁽¹⁰⁾,

$$U(r) = \frac{Z'^5}{24} \cdot r^4 \cdot e^{-Z'r}.$$

Then

$$\begin{aligned} F_0' &= \int_0^\infty U(r) \cdot \frac{\sin \phi}{\phi} \cdot dr, \\ &= \frac{Z'^5}{24k} \int_0^\infty r^3 \cdot \sin kr \cdot e^{-Z'r} \cdot dr, \end{aligned}$$

where

$$k = \frac{\phi}{r} = 4\pi \left(\frac{\sin \theta}{\lambda} \right).$$

On evaluating the integral

$$F_0' = \frac{1 - \left(\frac{k}{Z'} \right)^2}{\left[1 + \left(\frac{k}{Z'} \right)^2 \right]^4}$$

Z' was obtained by making F_0' , the hydrogen-like function,

* F_0' is the scattering power for a hydrogen-like distribution of charge with nuclear charge $Z'e$.

and F_0 , the Hartree function, coincide at $F_0=0.5$; that is to say, Z' is given by

$$\left[1 + \left(\frac{k}{Z'}\right)^2\right]^4 = 2 \left[1 - \left(\frac{k}{Z'}\right)^2\right].$$

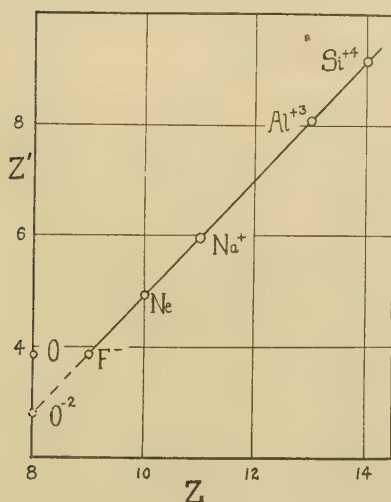
A graphical solution of this equation gives

$$\frac{k}{Z'} = 0.716,$$

whence

$$Z' = 17.5 \left(\frac{\sin \theta}{\lambda} \right)_{F_0=0.5},$$

Fig. 1.



The variation of Z' , the effective nuclear charge for X-ray scattering, with Z , the true nuclear charge, for the 22 electrons of neon-like ions.

where $\left(\frac{\sin \theta}{\lambda}\right)_{F_0=0.5}$ is the value of $(\sin \theta)/\lambda$, (λ in Å.U.), for which the Hartree F_0 function is 0.5.

The results obtained are shown in fig. 1, where Z' is plotted against Z ; Z' may be called the effective nuclear charge for X-ray scattering. The linearity of the variation of Z' with Z is clearly seen. It is of interest to observe how far the value of Z' for neutral oxygen lies from this line.

Another method by which F_0 was estimated for O^{-2} will be described fully in a later paper, but the principle of the

method is as follows. When F_0 for a particular group of electrons is plotted as a function of $\frac{1}{Z-s} \left(\frac{\sin \theta}{\lambda} \right)$, where Ze is the true nuclear charge, F_0 is practically independent of Z when s has the appropriate value. Values of s have now been obtained for the different groups of electrons in many atoms and ions. It is possible in the case of O^{-2} to estimate s with a considerable degree of certainty, and hence to calculate F_0 for O^{-2} .

Other methods have also been used, but they all lead to essentially the same result for F_0 . This may be taken as evidence that in estimating the F_0 curve for O^{-2} from data for other neon-like ions no serious errors have been made. The error in F_0 is probably nowhere greater than 0.1, and is probably less for those parts of the curve which are not steep with respect to the $(\sin \theta)/\lambda$ axis.

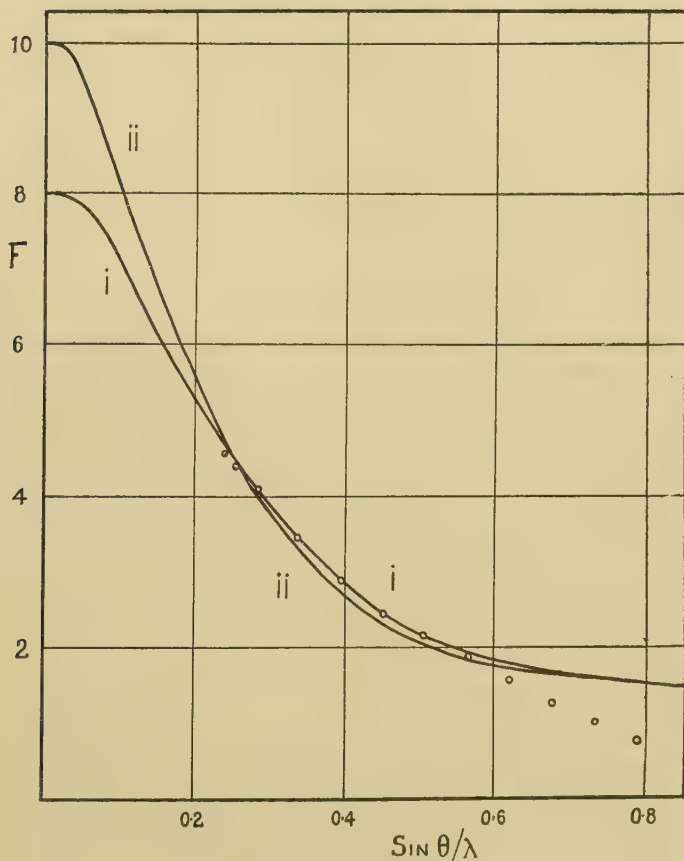
5. *A Discussion of the Results for Oxygen in MgO.*

In fig. 2 the theoretical F_0 curves are drawn for neutral O and for O^{-2} . At small values of $(\sin \theta)/\lambda$ the curves are quite different, owing to the number of electrons being different in the two cases. At large values of $(\sin \theta)/\lambda$ the curves are practically coincident; this is due to the fact that only the 1i electrons scatter appreciably at large angles, and the 1i electrons in neutral O and O^{-2} have the same distribution of charge. The most interesting region is between $(\sin \theta)/\lambda = 0.25$ and 0.65, where the two curves are slightly different. The physical significance of this result appears to be as follows. When two additional electrons are added to neutral oxygen to form O^{-2} there is an interaction between these electrons and those already present. Neutral oxygen is considered to have four 22 electrons, and O^{-2} , being neon-like, to have six 22 electrons. The interaction, therefore, of the two additional electrons in O^{-2} affects mainly the 22 group; owing to their mutual repulsion, the 22 electrons in O^{-2} have a more diffuse distribution of charge than the 22 electrons in neutral O. This may also be seen from the data given in fig. 1, Z' for neutral O being 3.86 and for O^{-2} 2.80. Hence, although the scattering power of O^{-2} tends to be greater, owing to its greater number of electrons, the effect is more than counterbalanced by the greater diffuseness of the electrons in O^{-2} , and, except at small values of $(\sin \theta)/\lambda$, F_0 for O^{-2} is slightly *smaller* than F_0 for neutral oxygen. The curves, however, are for the most part so nearly

coincident that it will be difficult to decide whether oxygen is ionized or not in MgO.

As stated above, before a comparison can be made between theory and experiment an estimate should be made of the temperature factor e^{-M} and the corresponding F_T curves

Fig. 2.



X-ray scattering curves for (i) neutral oxygen and (ii) O^{-2} . The circles show the experimental values of Wyckoff and Miss Armstrong.

calculated. However, it is seen from fig. 2 that there is a close agreement between the theoretical F_0 curves and the experimental results of Wyckoff and Miss Armstrong, which are shown in the figure by circles.

The fact that the agreement between the experimental results and the theoretical F_0 curves is so close without the application of a temperature correction indicates that the amplitude of thermal vibration of the oxygen atoms (or ions) in MgO is very small. This result is in accord with the general physical properties of MgO. Wyckoff and Miss Armstrong used artificial crystals, periclase, and in this form MgO is very hard. On Moh's scale of hardness periclase has a value approximately 6, a value which lies between the hardness of felspar and that of quartz. Other physical properties show that the atoms in MgO are very tightly bound together. For example, the fusion point is of the order of 2000°C . The coefficient of thermal expansion ($\alpha=0.0000114$) is not much greater than that of quartz. The compressibility also is very small. In such a crystal it is not surprising to find that the oxygen atoms have a small amplitude of thermal vibration.

Since the theoretical curves are so close together in the region of $(\sin \theta)/\lambda$ where experimental measurements have been made, it is difficult to draw a definite conclusion as to the state of ionization of the oxygen atoms. Between $(\sin \theta)/\lambda=0.3$ and 0.6 the experimental results certainly fit the neutral oxygen curve better than the O^{-2} curve. Moreover, if some small allowance be made for thermal vibration, then it is possible to fit the neutral O curve *slightly better* to the experimental results, but any such allowance increases the difference between the O^{-2} curve and the experimental results. The closest agreement is obtained between the theoretical F_T curve for neutral oxygen and the experimental data when

$$F_T = F_0 \cdot e^{-0.20 (\sin^2 \theta)/\lambda^2},$$

$$\text{i.e., } \alpha=0.20, \text{ whence } \sqrt{u^2}=0.087 \text{ A.U.}$$

We may therefore draw the following conclusion. The experimental results are in agreement with the view that oxygen in MgO exists as neutral oxygen rather than as O^{-2} , and a comparison of theory with experiment points to neutral O rather than to O^{-2} as existing in MgO, but in view of the small difference between the theoretical results for neutral O and O^{-2} an unquestionable decision can hardly be given. This is in agreement with the result obtained by Wollen⁽⁴⁾ using Fourier analysis. The present method shows clearly the uncertainty attached to the determination of the number of electrons in atoms from X-ray scattering data; in the analytical method there is also uncertainty, but in general it is not so evident.

There is one further point of interest. Beyond $(\sin \theta)/\lambda = 0.6$ the experimental values fall rapidly. It is very difficult to find an explanation for this result, but it seems to be connected in some way with the use of *powdered* crystals. In the experiments of James single crystals were used, and although the measurements were made at *larger* values of $(\sin \theta)/\lambda$ than are used with powdered crystals, there was no sign of any sudden falling off of F to zero at large $(\sin \theta)/\lambda$. Further, the results of James were in close agreement with theory. Attention has recently been directed to this point ⁽²⁾, for similar measurements by R. J. Havighurst ⁽¹¹⁾ with powdered crystals show the same effect.

6. The Scattering Powers of Neutral Magnesium and of Mg^{+2} .

The scattering powers of Mg and of Mg^{+2} have been calculated by the method referred to in section 4, in which F_0 is regarded as a function of $\frac{1}{Z-s} \left(\frac{\sin \theta}{\lambda} \right)$. The results are shown in fig. 3, and, as anticipated earlier, the curves do not differ except at very small values of $(\sin \theta)/\lambda$. The experimental results are shown in the figure by circles. No information concerning the state of ionization of magnesium can be obtained from this figure. It is of interest to note, however, that the magnesium atom (or ion) appears to have a much larger amplitude of thermal vibration than the oxygen atom. The closest agreement between F_T , the theoretical scattering curve for Mg (or Mg^{+2}), and the experimental data is obtained by taking

$$F_T = F_0 e^{-0.33 (\sin^2 \theta)/\lambda^2},$$

$$i. e., \quad \alpha = 0.33, \quad \text{whence} \quad \sqrt{\overline{u^2}} = 0.11_2 \text{ \AA.U.}$$

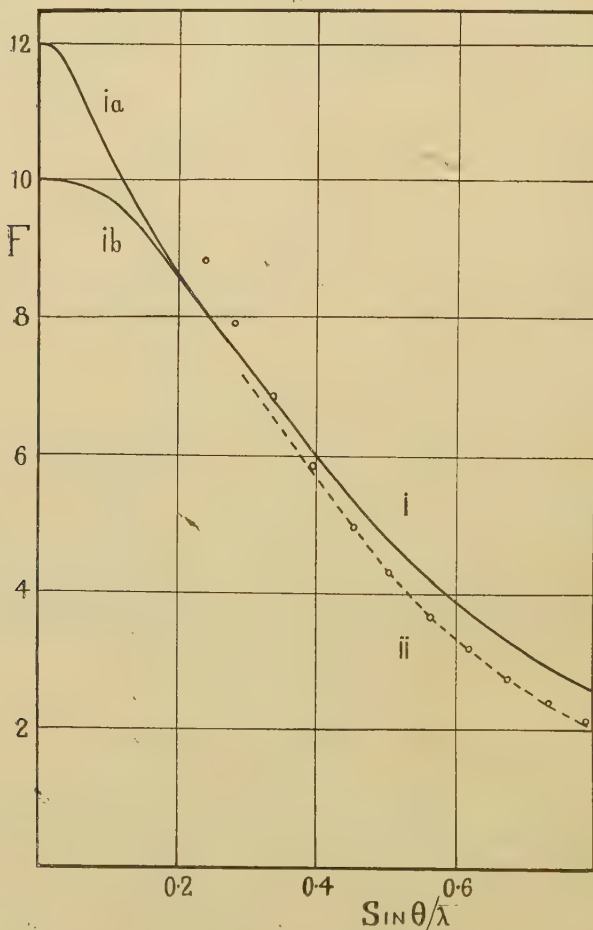
The difference between the estimated amplitudes of vibration, 0.11 \AA.U. for Mg and 0.087 \AA.U. for O , is not so great as might have been expected from a glance at figs. 2 and 3. However, since

$$\overline{u^2} = \frac{3\alpha}{8\pi^2} = \frac{5}{8\pi^2} \cdot \frac{\log_e(F_0/F_T)}{(\sin^2 \theta)/\lambda^2},$$

the amplitude of vibration, $\sqrt{\overline{u^2}}$, depends on the ratio (F_0/F_T) ; F_0 and F_T have a larger absolute difference in the case of magnesium, but this is counterbalanced by F_0 and F_T each being about twice as big for Mg as for O , for $(\sin \theta)/\lambda > 0.3$. The values given above for the amplitudes of vibration must only be regarded as *rough estimates*

for the difference between F_T and F_0 , particularly in the case of oxygen, is too small for accurate values to be obtained. It is, however, worth noting that in MgO , as in $NaCl$ and LiF , the smaller ion has the larger amplitude of vibration.

Fig. 3.



X-ray scattering curves for (i a) neutral Mg and (i b) Mg^{+2} . Curve (ii) is given by $F_T = F_0 e^{-0.33 (\sin^2 \theta) / \lambda^2}$. The circles show the experimental results of Wyckoff and Miss Armstrong.

In conclusion, I would like to take this opportunity to thank Professor Hartree for his kindness in allowing me to make use of his results for some of the neon-like ions.

7. Summary.

(i.) It is pointed out that the determination of the number of electrons associated with the atoms of magnesium and oxygen in MgO by means of Fourier analysis is not practicable owing to the uncertainties inherently connected with the method.

(ii.) An alternative method is suggested based on the principle that the scattering power of an atom for X-rays can be calculated, knowing the distribution of charge in the atom and a temperature factor to correct for the thermal vibration.

(iii.) Scattering curves are given for neutral O, O^{-2} , neutral Mg, and Mg^{+2} . Certain difficulties in the case of O^{-2} were surmounted by a method of extrapolation.

(iv.) In the case of oxygen, the scattering curves for O and O^{-2} are very similar, and there is good agreement with the experimental results. The data are in keeping with the view that the oxygen atom in MgO is un-ionized and, in fact, point to this conclusion, but the scattering curves for O and O^{-2} are not sufficiently different for a decisive answer to be given.

(v.) In the case of magnesium, no information can be obtained concerning the state of ionization of the atom.

(vi.) Estimates are made of the amplitudes of vibration of the atoms in the lattice, the results being 0.11 Å.U. for Mg and 0.09 Å.U. for O, the smaller ion having the larger vibration as in NaCl and LiF, but these values can only be regarded as *rough estimates*. Such small values for the amplitudes of vibration are in accord with the general physical properties of crystalline MgO.

8. References.

- (1) R. W. G. Wyckoff and A. H. Armstrong, *Zeit. f. Krist.* lxxii. p. 433 (1930).
- (2) G. W. Brindley, *Phil. Mag.* ix. p. 193 (Feb. 1930).
- (3) A. H. Compton, see 'X-rays and Electrons,' ch. v.
- (4) E. O. Wollen, *Phys. Rev.* xxxv. p. 127 (1920).
- (5) I. Waller and R. W. James, *Proc. Roy. Soc. A*, cxvii. p. 214 (1927); R. W. James, I. Waller, and D. R. Hartree, *Proc. Roy. Soc. A*, cxviii. p. 334 (1928); R. W. James and G. W. Brindley, *Proc. Roy. Soc. A*, cxxi. p. 155 (1928).
- (6) See I. Waller and R. W. James, *loc. cit.*
- (7) R. W. James, G. W. Brindley, and R. G. Wood, *Proc. Roy. Soc. A*, cxxv. p. 401 (1929).
- (8) D. R. Hartree, *Proc. Camb. Phil. Soc.* xxiv. pp. 89, 111 (1928).
- (9) J. A. Gaunt, *Proc. Camb. Phil. Soc.* xxiv. p. 328 (1928); J. C. Slater, *Phys. Rev.* xxxii. p. 343 (1928); also xxxv. p. 211 (1930).

(10) See E. Schrödinger, *Ann. d. Physik*, lxxix. p. 361 (1926), see equation (18), p. 369.

(11) R. J. Havighurst, *Phys. Rev.* xxviii. p. 869 (1926).

The Physical Laboratories,
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March 14, 1930.

Note added in proof, April 26th.—In a private communication Prof. Hartree has pointed out to me that a consideration of other physical properties of MgO indicates that the atoms in the crystal are ionized; *e.g.* NaF and MgO have the same structure, but AlN is different, which suggests that NaF and MgO are both formed of ionized atoms, but that AlN is different; the interatomic distance also agrees with Lennard-Jones's calculated value based on Mg^{+2} and O^{-2} (*Proc. Roy. Soc.* cix. p. 476, table vii.). If the atoms in MgO are really ionized, then it would seem that reliable information about the state of ionization of atoms in crystals cannot be obtained from X-ray scattering curves. Uncertainty arises both in the analytical method and in the alternative method discussed above owing to the difficulty of obtaining sufficiently extensive experimental data.

CI. *Problems of determining Initial and Maximum Stresses in Ties and Struts under Elastic or rigid End Constraints.*—
Part III.* *By* W. H. BROOKS, *B.Sc., Ph.D.(Eng.) Lond.*†
To establish Expressions for Y/X for Struts flexed by Various Methods.

THE strut equations are quickly derived from the tie results by substituting in the latter $-P$ for P , and therefore $i.n$ for n , where $i = \sqrt{-1}$; remembering that $n = \sqrt{P/EI}$, and that

$$\sinh i.n.l = i.\sin nl.$$

$$\cosh i.n.l = \cos nl.$$

$$\tanh i.n.l = i.\tan n.L.$$

Strut Method 1A. See fig. 4 (vol. viii. p. 947, 1929), with P reversed:—

From the tie equation (4) for deflexion δ at C,

$$\delta/W = -(L - \tanh inL/in)/2P = -(L - i.\tan nL/in)/2P$$

$$= -(L - \tan nL/n)/2P = (\tan nL/n - L)/2P. \quad (70)$$

* See Part I., *Phil. Mag.* (7) viii. p. 943 (1929); and Part II., *Phil. Mag.* (7) ix. p. 426 (1930). Figs. 1-11 and Charts I.-VI. (referred to in this Part) appeared in Parts I. and II.

† Communicated by the Author.

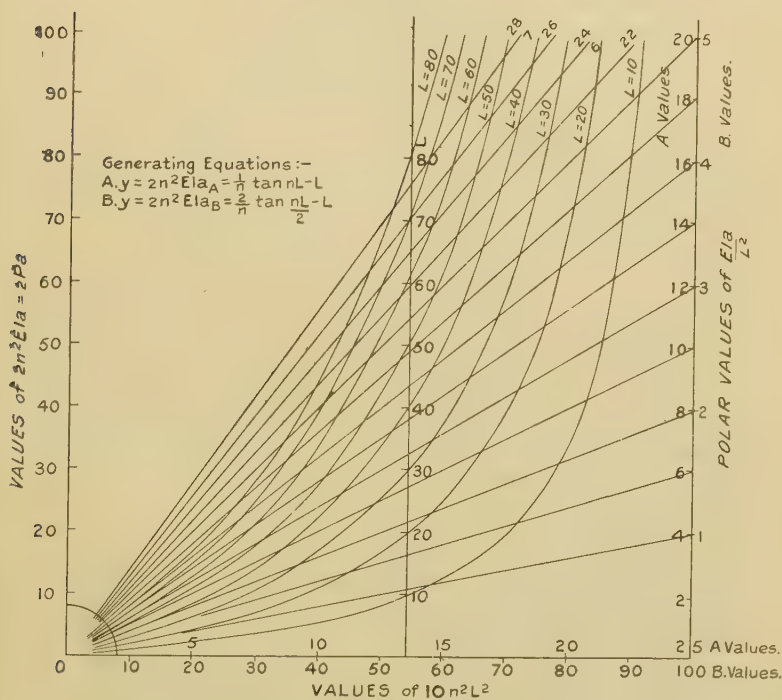
That is, initially, when $\delta = 0$ and $W = 0$,

$$“a” = (\tan n_a L_a / n_a - L_a) / 2n_a^2 \cdot EI. \quad (71)$$

(Strut equation A δ .)

In equation (71), “ a ” is the $W.\delta$. derivative, and is found experimentally as described in the general procedure. Having found “ a ,” the solution to $n_a^2 I_a^2$, and thence the solution to P_a , i. e., $n_a^2 EI$, may be readily obtained from

CHART VII.



Stress Chart for Struts A and B by deflexion.

Chart VII., on which the suffix a to n and to L is suppressed for reasons of clarity. To obtain a solution, the value of EIa/L^2 is located on the vertical scale of A values—drawn in where $10n^2L^2 = 25$,—and a polar ray is drawn through the origin of the chart. The curve of L (the half-strut length) is next interpolated by aid of the vertical scale of L drawn in where $10n^2L^2 = 13.5$. The projection on to the abscissæ of the point of intersection of this latter curve with the former ray gives the solution sought.

Applying the expansion

$$\tan \beta = \beta + \beta^3/3 + 2\beta^5/15 + 17\beta^7/315 + \dots$$

to equation (70), and writing nL for β , it becomes

$$\delta/W = L(1 + n^2L^2/3 + 2n^4L^4/15 + 17n^6L^6/315 + \dots - 1)/2P,$$

and substituting

$$n^2 = \pi^2 P/4L^2 P_e,$$

$$\text{where } P_e = \pi^2 EI/4L^2,$$

the Euler buckling value of P ,

$$\begin{aligned} \delta/W &= L\{\pi^2/12 \cdot (P/P_e) + \pi^4/120 \cdot (P/P_e)^2 \\ &\quad + 17\pi^6/20160 \cdot (P/P_e)^3 + \dots\}/2P \\ &\doteq 2L\{(P/P_e) + (P/P_e)^2 + (P/P_e)^3 + \dots\}/5P. \end{aligned}$$

The latter expression in brackets is now a G.P. having a common ratio $= P/P_e$ less than unity, and may therefore be summed to infinity. Thus

$$S_\infty = P/(P - P_e),$$

giving

$$\delta/W \doteq 2L/5(P_e - P), \quad (72)$$

or, initially, " a " $= 2L/5(P_e - P_a)$;

$$\therefore P_a \doteq P_e - 2L/5a. \quad (73)$$

This last expression may therefore be used for an approximate solution in this case, or as a first approximation when seeking for an exact solution to equation (71).

For a *strut* therefore, tested as in this case,

$$2L/5a < P, \quad \text{i.e., } 2L/a < 5\pi^2 EI/4L^2$$

approximately ; and this result may be used to check a doubtful case of stress reversal (see section on Stress Reversal), for when $2L/5a$ is found to be $> P_e$, i.e., when " a " is found to be $< 8L^3/5\pi^2 EI$, P_a is shown to be negative, and therefore the member is a *tie*.

Conversely, when " a " is found to be $> 8L^3/5\pi^2 EI$, the member is a *strut*.

In other words, when $P_a = 0$, the above approximation shows that the value of " a " (which would then be constant and $= \delta/W$),

$$\doteq 8L^3/5\pi^2 EI. \quad (74)$$

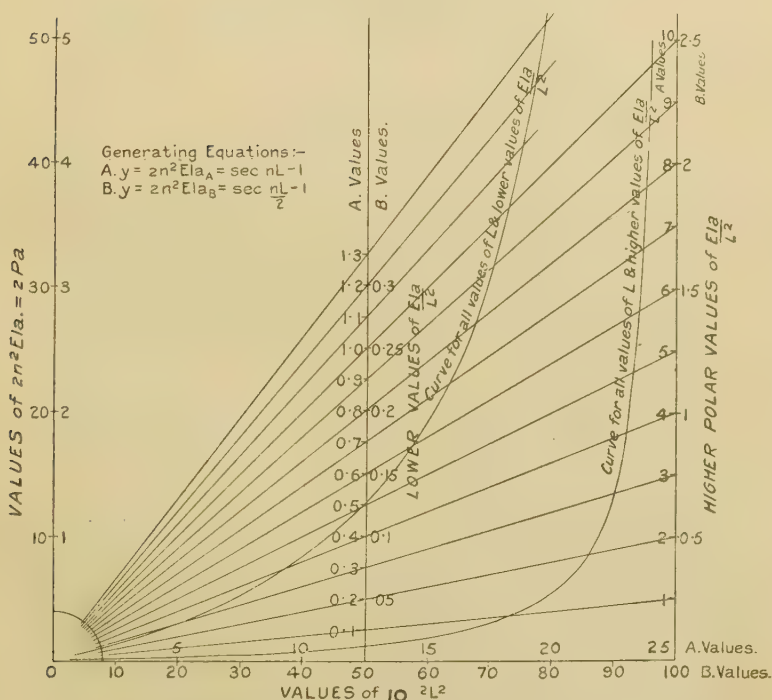
The exact value of this ratio for the above case—as may readily be deduced for the equivalent simply supported beam—is $L^3/6EI$, and should preferably be used in a closer check for stress reversal. For any case the value of " a "

when $P = 0$, will hereinafter be referred to as the "reversal ratio," and is fully discussed in the section on Reversal Ratios, later.

Strut Method 1 A. Alternatively by slope. See fig. 5 (vol. viii. p. 950, 1929), with P reversed:—

From the tie equation (8) for the slope θ at A or B,
 $\theta/W = -(1 - 1/\cosh nL)/2P = -(1 - 1/\cos nL)/2P$
 $= -(1 - \sec nL)/2P = (\sec nL - 1)/2P, \quad (75)$

CHART VIII.



Stress Chart for Struts A and B by slope.

or, initially, when $\theta = 0$ and W also $= 0$,

$$"a" = (\sec n_a L_a - 1)/2n_a^2 \cdot EI, \quad (76)$$

(Strut equation A θ)

where "a" is the $W.\theta$ derivative.

Graphical solutions to n_a^2 in this equation may be quickly obtained from the A values of Chart VIII., which is used in

a similar way to that described for Chart VII. Here also the suffix a is suppressed for reasons of clarity.

Applying the expansion

$$\sec \beta = 1 + \beta^2/2! + 5\beta^4/4! + 61\beta^6/6! + \dots$$

to equation (75), it becomes

$$\theta/W = (n^2 L^2/2! + 5n^4 L^4/4! + 61n^6 L^6/6! + \dots);$$

and substituting $n^2 L^2 = \frac{\pi^2}{4} \cdot \frac{P}{P_e}$,

it becomes

$$\begin{aligned} & \{ \pi^2/8 \cdot (P/P_e) + 5\pi^4/16 \cdot 4! \times (P/P_e)^2 \\ & \quad + 61\pi^6/64 \cdot 6! \times (P/P_e)^3 + \dots \} / 2P \\ & \doteq 5/8P \cdot \{ (P/P_e) + (P/P_e)^2 + (P/P_e)^3 + \dots \}; \end{aligned}$$

and, summing to infinity as before,

$$\theta/W \doteq 5/8(P_e - P), \quad . \quad . \quad . \quad . \quad (77)$$

or, initially, $P_a \doteq P_e - 5/8a, \quad . \quad . \quad . \quad . \quad (78)$

and may be used as a first approximation when seeking a solution to equation (76).

Equation (78) shows that in this case, where the member is a *strut*, approximately, $5/8a$ must be $< P_e$, i. e., $< \pi^2 EI/4L^2$, or " a " $> 5L^2/2\pi^2 EI$, and, when $P_a = 0$, " a " $\doteq 5L^2/2\pi^2 EI$.

When P is reversed, the member becomes a *tie*, and then " a " $< 5L^2/2\pi^2 EI$.

The exact value of this reversal ratio, i. e., the value of the $W.\theta$. derivative when $P = 0$, as deduced from the equivalent simply supported beam for this case, is $L^2/4EI$, and should preferably be used in a closer check.

That is, the member is a $\left\{ \begin{array}{c} \text{strut} \\ \text{or} \\ \text{tie} \end{array} \right\}$ according as the initial $W.\delta$. derivative $\left\{ \begin{array}{c} > \\ \text{or} \\ < \end{array} \right\} L^3/6EI$ (fig. 4), or according as the initial $W.\theta$. derivative $\left\{ \begin{array}{c} > \\ \text{or} \\ < \end{array} \right\} L^2/4EI$ (fig. 5) (see later section on Stress Reversal and Reversal Ratios).

Strut Method 1 B. See fig. 6 (vol. viii. p. 952, 1929), with P reversed:—

From the tie equation (14) for the *deflexion* δ at C ,

$$\begin{aligned} \delta/W &= -\{L - 2/in. \tanh(inL/2)\}/2P \\ &= -\{L - 2/in. i. \tan(nL/2)\}/2P \\ &= -(L - 2/n. \tan nL/2)/2P = (2/n. \tan nL/2 - L)/2P. \quad (79) \end{aligned}$$

That is, initially,

$$“a” = (2/n_a \cdot \tan \overline{n_a L_a/2} - L_a)/2 \cdot n_a^2 EI, \dots (80)$$

(Strut equation B_δ)

where “a” is the W.δ. derivative.

Graphical solutions to n_a^2 in this equation may be readily obtained from the B values of Chart VII. in a similar way to that described for Strut Method 1A.

Expanding equation (79) similarly to the expansion of equation (70), it becomes

$$\delta/W = L/2P \cdot (1 + n^2 L^2/12 + n^4 L^4/120 + 17n^6 L^6/20160 + \dots - 1),$$

and substituting $n^2 L^2 = \pi^2 \cdot P/P_e$, where $P_e = \pi^2 EI/L^2$ in this case,

$$\begin{aligned} \delta/W &= L/2P \cdot \{ \pi^2/12 \cdot (P/P_e) + \pi^4/120 \cdot (P/P_e)^2 \\ &\quad + 17/20160 \cdot (P/P_e)^3 + \dots \} \\ &\doteq 2L/5P \{ (P/P_e) + (P/P_e)^2 + (P/P_e)^3 + \dots \}, \end{aligned}$$

as in Method 1A,—but with a different value for P_e , and therefore yielding the same equations (72) and (73) in P_e , to which corresponding deductions apply.

Hence, for a *strut* with clamped ends as in this case, the value of the initial W.δ. derivative must approximately be $> 2L^3/5\pi^2 EI$, i. e., $> \frac{1}{4}$ the approximate value obtained in Method 1A.

When “a” is found to be approximately $< 2L^3/5\pi^2 EI$, the member is shown to be a *tie*.

When $P_a = 0$,

$$\text{the constant value of “a”} = \delta/W \doteq 2L^3/5\pi^2 EI. \quad (81)$$

Here the exact value of the reversal ratio, as deduced from the equivalent encasté beam with $P = 0$, is $L^3/24EI$, or $\frac{1}{4}$ the reversal ratio of Method 1A, and should be used when a closer check is required. (See later section on Reversal Ratios.)

Strut Method 1B. Alternatively by slope θ at $\frac{1}{4}$ span.

See fig. 7 (vol. viii. p. 952, 1929), with P reversed:—

From the tie equation (18),

$$\theta/W = -(1 - \operatorname{sech} \overline{inL/2})/2n^2 EI = (\sec \overline{nL/2} - 1)/2n^2 EI, \dots (82)$$

$$\text{or, initially, “a”} = (\sec \overline{n_a L_a/2} - 1)/2n_a^2 EI. \dots (83)$$

(Strut equation B_θ)

Graphical solutions to n_a^2 in this equation may readily be obtained from the B values of Chart VIII. in a similar way to that previously described for alternative Method 1 A.

Expanding equation (83) similarly to the expansion of equation (75), it becomes on substituting for P_e , which $= \pi^2 EI/L^2$,

$$“a” \doteq 5/8 P_a \cdot \{ (P_a/P_e) + (P_a/P_e)^2 + (P_a/P_e)^3 + \dots \};$$

and, summing to infinity as before and making corresponding deductions, we get as approximate relationships

$$5/8 a < P'_e, \quad i. e., < \pi^2 EI/L^2,$$

i. e., approximately the W. θ . derivative $> 5L^2/8\pi^2 EI$. That is, the approximate reversal ratio in this case is $\frac{1}{4}$ of the corresponding ratio obtained by Method 1 A. Its exact value is $L^2/16EI$, as may readily be deduced from first principles when $P = 0$.

Hence a member with clamped ends is a $\left\{ \begin{array}{c} \text{strut} \\ \text{or} \\ \text{tie} \end{array} \right\}$ according

as the initial W. δ . derivative $\left\{ \begin{array}{c} > \\ \text{or} \\ < \end{array} \right. L^2/24EI$ (fig. 6), or accord-

ing as the initial W. θ . derivative $\left\{ \begin{array}{c} > \\ \text{or} \\ < \end{array} \right. L^2/16EI$ (fig. 7).

Strut Method 2A. See fig. 8 (vol. ix. p. 426, 1930), with P reversed :—

From the tie equation (29) for the deflexion δ at C,

$$\begin{aligned} \delta/W &= -[l - \{ \sinh inl - \tanh inL (\cosh inl - 1) \} / in] / 2P \\ &= -[l - \{ i \cdot \sin nl - i \cdot \tan nL (\cosh nl - 1) \} / in] / 2P \\ &= [\{ \sin nl - \tan nL (\cos nl - 1) \} / n - l] / 2P. \quad . \quad . \quad (84) \end{aligned}$$

That is, initially, when $\delta = 0$ and W also $= 0$,

$$“a” = [\{ \sin n_a l + \tan n_a L \quad a l \} / n_a - l] / 2n_a^2 EI, \quad . \quad . \quad (85)$$

(Strut equation G δ)

where “a” = the initial W. δ . derivative.

Solutions to n_a^2 in equation (85) for an instrument constant $l = 10$ units may be readily obtained from the G values of Chart IX. by finding the point of intersection of the appropriate polar ray of $EI \cdot a/L^2$ with the correct curve of L , the length of the half-strut, which latter may be interpolated from the typical curves given. Here, as on

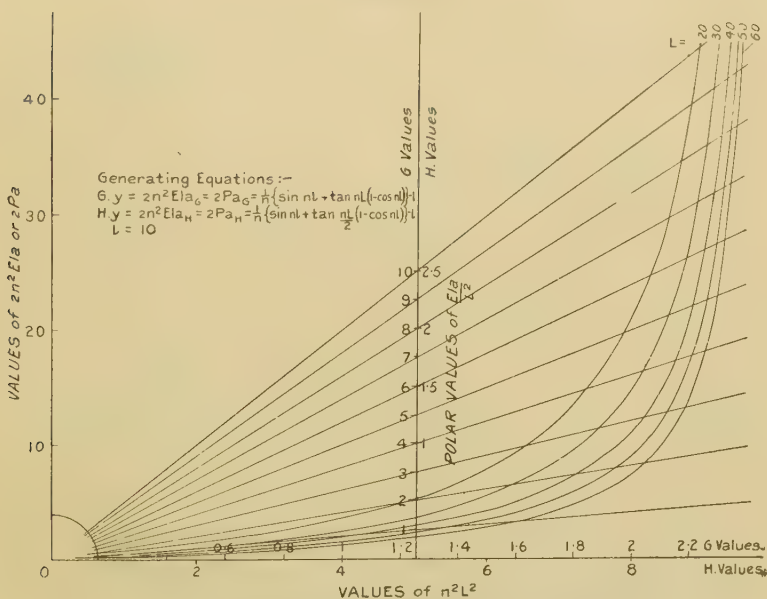
the other charts, the suffix a is suppressed for simplicity. Projecting from the point so found on to the abscissæ gives the solution to $P_a = n_a^2 \cdot EI$ sought.

Strut Method 2A. Alternatively by the slope θ at D :—

From tie equation (32),

$$\begin{aligned}\theta/W &= -(\cosh inl - 1)(\cosh inl - \sinh inl \cdot \tanh inL)/2P \\ &= -(\cos nl - 1)(\cos nl - i \cdot \sin nl \cdot i \cdot \tan nL)/2P \\ &= (1 - \cos nl)(\cos nl + \sin nl \cdot \tan nL)/2P, \dots (86)\end{aligned}$$

CHART IX.



Stress Chart for Struts G and H by deflexion.

or, initially,

$$\begin{aligned}“a” &= (1 - \cos n_a l)(\cos n_a l + \sin n_a l \cdot \tan n_a L_a)/2n_a^2 EI, \\ &\dots (87) \\ &\text{(Strut equation } G_\theta)\end{aligned}$$

where “ a ” is the initial $W \cdot \theta$ derivative.

Equation (87) is the generating equation of the G curves and values of Chart X. from which solutions to n_a^2 and thence solutions to $P_a = n_a^2 EI$ may be obtained by interpolation in a similar way to that described for Chart IX.

Strut Method 2 B. See fig. 9 (vol. ix. p. 429, 1930),
with P reversed:—

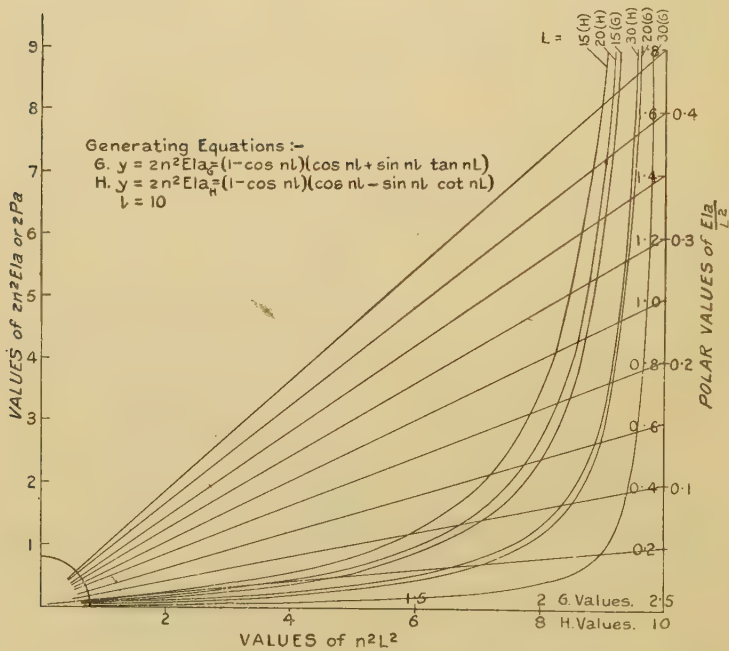
From the equation (41) for the deflexion δ at C,

$$\begin{aligned}\delta/W &= -[l - \{i \cdot \sin nl - i \cdot \tan \overline{nL/2} (\cos nl - 1)\} / in] / 2P \\ &= [\{\sin nl + \tan \overline{nL/2} (1 - \cos nl)\} / n - l] / 2P, \quad \dots (88)\end{aligned}$$

or, initially,

$$\begin{aligned}“a” &= [\{\sin n_a l + \tan \overline{n_a L_a/2} (1 - \cos n_a l)\} / n_a - l] / 2n_a^2 EI, \quad (89) \\ &\quad \text{(Strut equation } H_\delta)\end{aligned}$$

CHART X.



where “a” is the $W \cdot \delta$. derivative.

Solutions to this equation are given by the H values of Chart IX. to be used in a similar way to that described for Strut Method 2 A.

Alternatively, from the equation (42) for the slope θ at D,

$$\begin{aligned}\theta/W &= -(\cos nl - 1)(\cos nl - i \cdot \sin nl / i \cdot \tan nL) / 2P \\ &= (1 - \cos nl)(\cos nl - \sin nl \cdot \cot nL) / 2P, \quad \dots (90)\end{aligned}$$

or, initially, when both θ and $W = 0$,

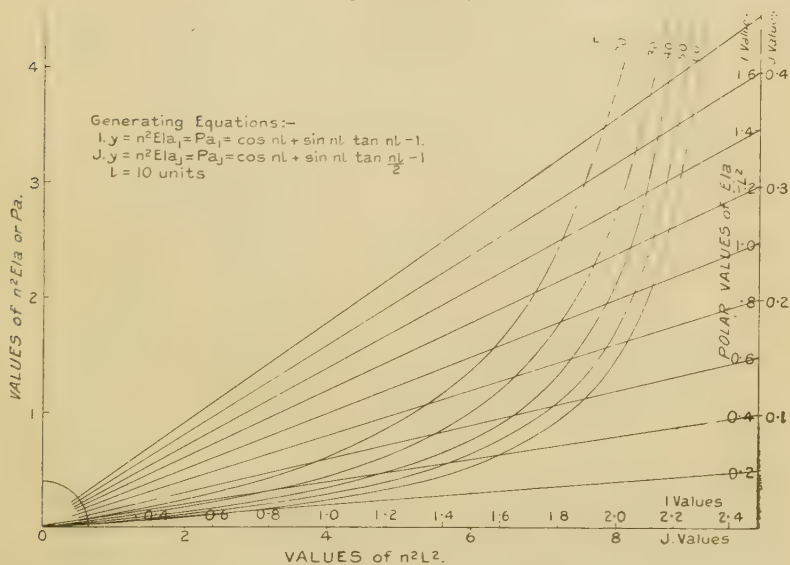
$$“a” = (1 - \cos n_a l) (\cos n_a l - \sin n_a l \cdot \cot n_a L_a) / 2n_a^2 EI, \quad (91)$$

(Strut equation $H\theta$)

where “ a ” in this case is the $W \cdot \theta$ derivative.

Solutions to equation (91) are given by the H values and curves of Chart X., which are used in a similar way to that described in the alternative Strut Method 2 A.

CHART XI.



Stress Chart for Struts I and J by deflexion.

Strut Method 3 A. See fig. 10 (vol. ix. p. 431, 1930), with P reversed:—

Here the ratio of the deflexion δ at C to M_1 at D is obtained from the corresponding tie equation (48), and is

$$\begin{aligned} \delta/M_1 &= -(1 - \cosh inl + \sinh inl \cdot \tanh inL)/P \\ &= -(1 - \cos nl - \sin nl \cdot \tan nL)/P \\ &= (\cos nl + \sin nl \cdot \tan nL - 1)/P; \dots \dots (92) \end{aligned}$$

and under initial conditions when $\delta = 0$ and M_1 also $= 0$,

$$“a” = (\cos n_a l - \sin n_a l \cdot \tan n_a L_a - 1)/n_a^2 EI, \quad (93)$$

where “ a ” here is the $M_1 \cdot \delta$ derivative and is obtained as described in the section on General Procedure.

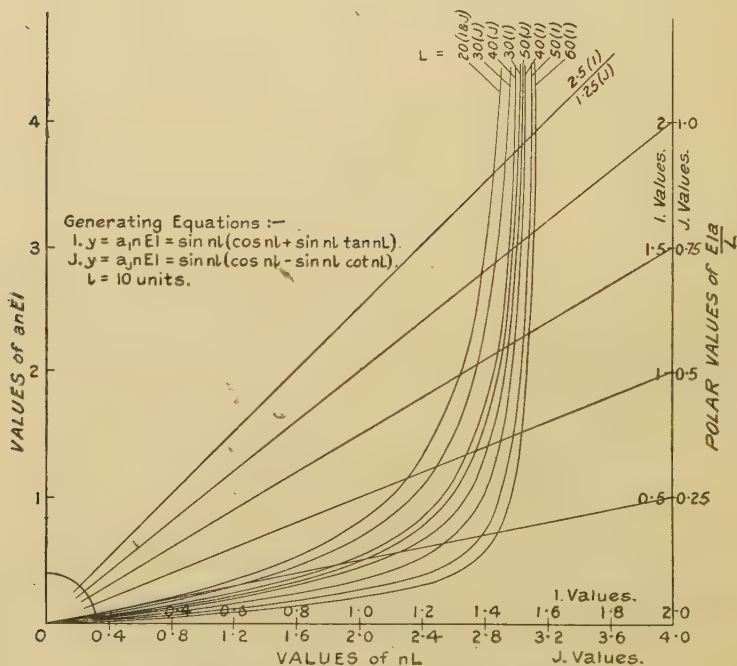
Solutions to n_a^2 in the above equation are given by the I values of Chart XI., which has been drawn for an instrument constant $l = 10$ units. The method of using this chart is similar to that previously described for the use of Chart IX. On this chart also the suffix "a" is suppressed.

Strut Method 3A. Alternatively from the ratio of the slope θ at D to the couple M_1 at D:—

From the equation (54),

$$\begin{aligned}\theta/M_1 &= i \cdot \sin nl / i \cdot n \cdot EI \cdot (\cos nl - i \cdot \sin nl \cdot i \cdot \tan nL) \\ &= \sin nl / n \cdot EI \cdot (\cos nl + \sin nl \cdot \tan nL), \quad . \quad . \quad (94)\end{aligned}$$

CHART XII.



Stress Chart for Struts I and J by slope.

or, initially,

$$"a" = \sin n_a l / n_a EI \cdot (\cos n_a l + \sin n_a l \cdot \tan n_a L_a), \quad (95)$$

(Strut equation I θ)

where "a" = the $M_1\theta$ derivative to be substituted, and is found as described in the "General Procedure."

The foregoing equation is the generating equation I of Chart XII.—to be used for its solution—upon which, as on the other charts, the suffix “ a ” is suppressed for clearness. To obtain a solution to n_a and hence a solution to $P_a = n_a^2 EI$, the 1 value of $EI.a/L$ on the ordinate drawn in at $nL = 2$ units on the abscissa is first located; then a polar ray is drawn to the origin through the point so found. Where this ray intersects, the I curve of L —which may be interpolated from the typical curves shown—locates a point vertically above a point on the abscissa scale of I values which gives the solution to P_a in terms of $n_a L_a$.

Strut Method 3 B. See fig. 11 (vol. ix. p. 435, 1930), with P reversed:—

Considering tie equation (63), which was established for the deflexion δ at C, the derived corresponding strut equation here required is

$$\begin{aligned}\theta/M_1 &= -(1 - \cos nl + i \cdot \sin nl \cdot i \cdot \tan \overline{nL/2})/P \\ &= (\cos nl + \sin nl \cdot \tan \overline{nL/2} - 1)/P, \quad . \quad . \quad (96)\end{aligned}$$

or, initially,

$$\begin{aligned}“a” &= (\cos n_a l + \sin n_a l \cdot \tan \overline{n_a L_a/2} - 1)/n_a^2 EI. \quad (97) \\ &\quad (Strut equation J_\delta.)\end{aligned}$$

Here “ a ” is the $M_1 \cdot \delta$ derivative, and is found as described in the “General Procedure.”

By substituting the value of “ a ” so found in equation (97), this equation may then be solved for n_a^2 , and thus $P_a = n_a^2 EI$ calculated, by the aid of the J values of Chart XI., which has been drawn for an instrument constant $l = 10$ units, and is used in a similar way to that described for the corresponding Method 3 A.

Strut Method 3 B. Alternatively by slope:—

From the tie equation (65) for the slope θ and couple M_1 at D, the ratio obtaining here is seen to be

$$\begin{aligned}\theta/M_1 &= i \cdot \sin nl / i \cdot nEI \cdot (\cos nl - i \cdot \sin nl / i \cdot \tan nL) \\ &= \sin nl (\cos nl - \sin nl \cdot \cot nL) / nEI. \quad . \quad . \quad (98)\end{aligned}$$

That is, initially, when both θ and $M_1 = 0$,

$$\begin{aligned}“a” &= \sin n_a l (\cos n_a l - \sin n_a l \cdot \cot n_a L_a) / n_a EI, \quad (99) \\ &\quad (Strut equation J_\theta)\end{aligned}$$

where “ a ” is the $M_1 \theta$ derivative. To effect a solution to this equation from any value of “ a ” experimentally deter-

mined, the J values and curves of Chart XII. should be used in a similar way to that described for Method 3A alternative.

An inspection of the foregoing equations for struts will show that, in the pin-jointed cases, the values of "a" become infinite when $n_a L_a = \pi/2$, and failure apparently occurs by flexing: i. e., for safety,

$$L_a(P/EI)^{\frac{1}{2}} \not\geq \pi/2, \text{ or } P_a \not\geq \pi^2 EI/4L_a^2,$$

which latter is the Euler buckling value.

Also, on inspecting the equations for struts with fixed ends, it will be seen that the values of "a" become infinite when $n_a L_a = \pi$, corresponding to the Euler buckling value of $\pi^2 EI/L_a^2$, as would be expected.

It must, however, be remembered, that in all cases the ends are elastically constrained, and, in the event of a strut commencing to buckle under a limiting end load, the initial compression in the strut will at once be reduced by the reactions of the end constraints as previously discussed.

It is also well known, and should here be borne in mind, that where initial eccentricity of P_a exists under weak constraints the critical values of $P_a <$ the Euler value P_e . The fact that P_a may not be truly axial to the strut does not however affect the relations given by the equations—a deduction fully discussed in the Thesis, of which this is an abstract.

Correction Factors for Non-central Loading of Struts by Methods 2 A and 2 B, 3 A and 3 B.

In both deflexion and slope investigations by the foregoing tie methods it is shown in the Thesis that a correction factor

$$= (1 \pm \frac{\tanh nL_1 - \tanh nL}{\tanh nL_1 + \tanh nL})$$

appears in each of the final expressions obtained for non-central loading, which factor practically reduces to unity when $nL > 3$, as is most likely to obtain, i. e. initially, when $n_a L_a > 3$. L and L_1 are the end distances from the non-central origin midway between the flexing loads or couples applied to the tie.

The corresponding correction factor applicable to struts is therefore

$$\begin{aligned} & (1 \pm \frac{\tanh inL_1 - \tanh inL}{\tanh inL_1 + \tanh inL}) \\ &= (1 \pm \frac{i \cdot \tan nL_1 - i \cdot \tan nL}{i \cdot \tan nL_1 + i \cdot \tan nL}) \\ &= (1 \pm \frac{\tan nL_1 - \tan nL}{\tan nL_1 + \tan nL}). \end{aligned}$$

Now this factor only reduces to unity when $nL = nL_1$, i. e., for central loading conditions only. Hence care must be exercised when making a strut test to ascertain that the flexing bridge or flexing couples are placed centrally, or, in particular cases where it is impossible to do so, the above correction factor must be introduced to obtain true results, using the stress charts as drawn to obtain first approximations only.

Obviously, the "a" expressions established by Methods 1A and 1B are only applicable to centrally tested cases—whether strut or tie—and so will not further be explored here.

Stress Reversal.

Where deformation of a structure has occurred through perhaps the subsidence of a support, the partial failure of a member or through other causes, cases may arise in which certain members, originally tensile, may suffer a reversal of stress, and so be called upon to function as struts, and *vice versa*. Now the former state of affairs may be serious on account of the smaller cross-sectional area to resist buckling, and, if the reversed stress in the tie is of sufficient magnitude to cause buckling, the state of affairs will, of course, be apparent on inspection. When, however, $-P_a < P_e$, mere inspection will furnish no evidence of stress reversal either qualitatively or quantitatively.

By applying the principles of the foregoing methods, choosing, say, two methods to check the results of each, it should be possible to obtain complete evidence in doubtful cases of this nature both as regards the sign of the stress present and also as regards its magnitude.

Criteria for stress reversal have already been noted in two cases, under the treatment for struts, tested by Methods 1A and 1B. A third case is fully discussed in the Thesis. Tests applicable to all cases will now be considered.

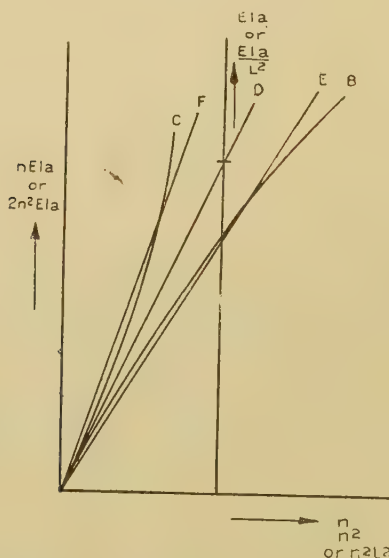
1. By the Reversal Ratio.

In the treatment by Methods 1A and 1B the term "reversal ratios" was introduced and explained in its natural sequence there. This term may be defined as "the value of the initial X.Y. derivative" when $P_a = 0$.

In the cases mentioned it was shown that a member is a *strut* or a *tie* according as the initial derivative $>$ the reversal ratio or $<$ the reversal ratio respectively. For reference purposes these reversal ratios are here tabulated with inclusion of the values for the remaining cases.

By method.	Deflexion ratio.	Fig. No.	Slope ratio.	Fig. No.
1 A	$\delta_C/W = L^3/6EI$	4	$\theta_D/W = L^2/4EI$	5
1 B	$\delta_C/W = L^3/24EI$	6	$\theta_D/W = L^2/16EI$	7
2 A	$\delta_C/W = l^2(3L-l)/12EI$	8	$\theta_D/W = l^2/4EI$	8
2 B	$\delta_C/W = l^2(3L-2l)/24EI$	9	$\theta_D/W = l^2(L-l)/4 \cdot EI \cdot L$	9
3 A	$\delta_C/M_1 = l(2L-l)/2EI$	10	$\theta_D/M_1 = l/EI$	10
3 B	$\delta_C/M_1 = l(L-l)/2EI$	11	$\theta_D/M_1 = l(L-l)/EI \cdot L$	11

Fig. 12.



2. From the Stress Chart for the Method used.

Where a stress chart is used the evaluation of the reversal ratio will be obviated, since a multiple of this ratio is given by the limiting tangent to the appropriate curve of L at the origin, and on most of the charts appears as the product of $EI \cdot a$ ("a" being the initial derivative when $P_a = 0$, and is consequently the reversal ratio).

Thus in fig. 12 a typical L curve OB from a tie chart, and a curve OC for the same value of L from a corresponding strut chart, are shown as though drawn to the same scale. When $P_a = 0$, the polar ray OD becomes the common tangent to both curves with its value indicated in terms of $EI.a$. If the member be a tie, its ray will be as OE, *below* OD, and consequently giving an intersection only on the tie curve OB. If the member be a strut, its ray will be as OF, *above* OD, so giving an intersection on the strut curve only.

That is, for any determined value of the initial derivative and given value of L only *one* intersection can be obtained, and a member is shown to be a tie *or* a strut according as the intersection is found on the appropriate tie chart or on the corresponding strut chart. If the tie chart yields no solution to a supposed tensile case, the member has suffered reversal of stress, and the magnitude of the stress obtaining can be found from the corresponding strut chart.

General Deductions and Conclusions.

This section is intended to form a summary to the foregoing investigations and conclusions thereto, and to draw further conclusions respecting the relative merits or limitations of adaptability—as the case may be—of the various methods discussed, and is therefore later divided for convenience of reference into two subsections: (*a*) dealing with the methods for the determinations of the stresses in ties, and (*b*) dealing with the similar methods for the determinations of the stresses in struts.

Values of EI.

Throughout this research it is very apparent that in any tie or strut investigation the degree of exactitude experimentally obtainable depends fundamentally upon knowing the value of the flexural rigidity (EI)—similarly to knowing the value of Young's Modulus (E) only, for the well-established methods of incremental stress determinations by the measurement of direct strains.

This quantity (EI) is therefore of fundamental importance, and, for the subsequent testing of a member, its value should be readily obtainable from specification and acceptance test data. When the factor E is unknown the usual handbook value for the material may be taken, and will give approximate results. It was found throughout the experimental work done and fully recorded in the complete Thesis that—broadly speaking—any percentage error in the value of E

taken gave approximately double that percentage error in the derivative " a ."

Here it should perhaps be pointed out that, although known relationships exist between E and other physical quantities of a material ^{(12), (13), (14)}, the direct determination of E from such quantities may not be possible when dealing with a member *in situ*, on account of the possible objection to the removal of a small quantity of the material for its determination. The other rigidity factor I , the second moment of area of the section, is, of course, a purely geometrical quantity, and so is obtainable from average cross-sectional measurements.

(a) Ties.

1. Central Single-load Method.

This method is obviously of limited application, and, although it is apparently the simplest of the three general methods discussed, it can only be easily applied to particularly placed members.

For ties of uniform cross-section to which central lateral loads may be conveniently applied as indicated in figs. 4 to 7 inclusive, the stresses present may be determined from

- (i.) the load-deflexion derivative of Method 1 A and Stress Chart I., A values ; or alternatively from
- (ii.) the load-slope derivative—where obtainable—from Method 1 A and Stress Chart II., A values, in smoothly pinned ended cases ; and from
- (iii.) the load-deflexion derivative of Method 1 B and Stress Chart I., B values ; or alternatively from
- (iv.) the load-slope derivative of Method 1 B and Stress Chart II., B values, in firmly fixed ended cases.

2. Flexing Bridge Method.

From the consistent experimental results obtained in the foregoing research, which results, together with the apparatus used, are fully recorded in the Thesis proper, it seems reasonable to infer that the Flexing Bridge Method—theoretically investigated by Methods 2 A and 2 B—will also give satisfactory experimental results, and may prove, as the analysis and resulting Stress Charts III. and IV. show, to have as wide and flexible an application as the following Couple Method, so that the former deductions of Method 1 should apply with equal verity to Methods 2 and 3 also.

3. *Couple Method.*

The stresses in ties to which it is not convenient or possible to apply independent lateral test loads may be determined by the application of simple flexing couples as indicated in figs. 10 and 11 (vol. ix. pp. 431, 435, 1930)—couples which may be applied anywhere along a continuous portion of the tie, irrespective of any tie joints on either side. The end conditions of fixity are immaterial in most actual cases.

For long or short ties of whatever inclination, flexible in any plane, the stresses present may be determined from

- (i.) the couple-deflexion derivative of Method 3 A and Stress Chart V. in smoothly pinned ended cases, or from
- (ii.) the couple-deflexion derivative of Method 3 B and Stress Chart V. in fixed ended cases.

Alternatively, the stresses present may be determined from

- (iii.) the couple-slope derivative of Method 3 A and Stress Chart VI. in smoothly pinned ended cases, or from
- (iv.) the couple-slope derivative of Method 3 B and Stress Chart VI. in fixed ended cases.

When in either Chart V. or Chart VI. the interpolated stress points lie on or near the dotted K curves, departures from the end conditions of fixity assumed are shown not to have affected the determinations.

(b) *Struts.*

1. *Central Single-load Method.*

For uniform smoothly pinned ended struts to which single lateral loads may be conveniently applied, as indicated in figs. 4 and 5 (with P reversed), first approximations to the stresses present may be determined from

- (i.) the load-deflexion derivative of Method 1 A and Stress Chart VII., A values; or alternatively from
- (ii.) the load-slope derivative—where obtainable—of Method 1 A and Stress Chart VIII., A values.

For uniform firmly fixed ended struts the stresses present may be determined, as indicated in figs. 6 and 7 (with P reversed), from

- (iii.) the load-deflexion derivative of Method 1 B and Stress Chart VII., B values; or alternatively from
- (iv.) the load-slope derivative—where obtainable—of Method 1 B and Stress Chart VIII., B values.

2. Flexing Bridge Method.

As concluded for ties, the similar consistent results obtained in this research on struts leads to the conclusion that the Flexing Bridge Methods 2 A and 2 B will also yield satisfactory experimental results if care be taken to ensure that the assumed theoretical conditions are provided for as closely as possible.

3. Couple Method.

In cases where it is not practicable to apply single lateral loads, the stresses in pinned ended struts having any inclination whatever may be determined by the application of flexing couples, as indicated in fig. 10 (with P reversed) from

- (i.) the couple-deflexion derivative of Method 3 A and Stress Chart XI., I values ; or alternatively from
- (ii.) the couple-slope derivative of Method 3 A and Stress Chart XII., I values.

For struts having fixed ends and flexible in whatever planes, the stresses present may be found, as indicated in fig. 11 (with P reversed), from

- (iii.) the couple-deflexion derivative of Method 3 B and Stress Chart XI., J values ; or alternatively from
- (iv.) the couple-slope derivative of Method 3 B and Stress Chart XII., J values.

The experimental part of this research, fully described in the Thesis, was carried out by the author in the Mechanical Laboratories of the Northampton Polytechnic Institute, London, during the four years 1925 to 1928, and he here desires to record his thanks to C. E. Larard, Esq., M.I.C.E., M.I.Mech.E., Head of the Civil and Mechanical Engineering Department, for his encouragement throughout its prosecution.

References.

- (12) "The Physical Properties of Metals as Functions of each other." By Dr. A. H. Stuart. *Journal of the Institute of Metals*, no. 2, 1915.
- (13) "The Modulus of Elasticity and its Relation to other Physical Quantities." By Dr. A. H. Stuart. *Proc. Inst. of Mech. Engineers*, December 1912.
- (14) "Relation between Young's Modulus, Density, and Atomic Weight." By T. Peczkalski. *Comptes Rendus*, clxxvi. pp. 500-502, Feb. 19, 1923.

CII. *A Hot-Wire Amplifier Method for the Measurement of the Distribution of Vortices behind Obstacles.* By E. TYLER, M.Sc., F.Inst.P., Lecturer, Physics Department, College of Technology, Leicester*.

INTRODUCTION.

THE desirability of using a hot-wire method for measuring the longitudinal and lateral spacing of vortices behind obstacles of small section, such as cylinders and plates, when placed in a steady air stream, necessitates greater sensitivity being obtained for the detection of the velocity fluctuations caused by the vortex formation than when using larger models. When an electrically heated platinum wire of small diameter is employed for the detection of vortex frequency, if the periodic heating and cooling of it occurs at a frequency of 100/sec. or more, the sensitiveness of it as a detector is considerably decreased. Moreover, when small models are used, even at moderately low air speeds, the frequency of formation of the vortices is generally well above the lower limit of audibility, and again a reduction in sensitivity is produced.

In previous papers†, I have described methods in which the longitudinal and lateral spacing of vortices formed behind cylinders at low frequencies have been determined, and the methods described herein are the results of an attempt to obtain data in a much shorter time for smaller models than in the previous methods, combined with increased sensitivity.

The investigations are classified under three headings:—

- 1) Measurement of the longitudinal spacing of vortices behind cylinders, using a differential hot-wire method.
- (2) Measurement of the lateral spacing of vortices behind cylinders.
- (3) Estimation of frequency formation of vortices behind cylinders, inclined plates, and aerofoils.

(1) *Longitudinal Spacing of Vortices behind Cylinders in Air. Differential Hot-wire Method.*

The arrangement for the determination of such spacing of vortices is as in fig. 1.

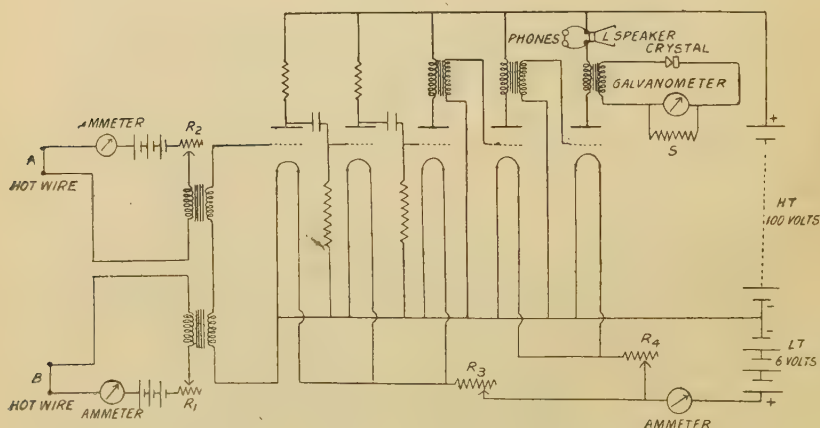
* Communicated by the Author.

† Journal of Scientific Inst. iii. No. 12, Sept. 1926; Phil. Mag. v. March 1928; Journal of Scientific Inst. vi. No. 10, Oct. 1929.

Two electrically heated platinum wires A and B, each of length 1 inch and diameter .001 inch, supplied with constant heating currents of .35 amp., formed part of two similar electrical circuits. Each wire was coupled to the grid input circuit of a five-valve amplifier by means of separate primaries of a transformer, and the plate circuit of the last valve was transformer-coupled to a circuit consisting of a crystal rectifier and a shunted reflecting galvanometer.

The wires were arranged similarly, as in the previous method, each mounted vertically at the ends of prongs of separate forks, and placed behind the cylinder in the middle of a wind channel ($1\frac{1}{2} \times 1\frac{1}{2} \times 15$ ft.), through which a steady stream of air flowed. Now, when each wire is suitably

Fig. 1.



placed, one in each vortex row, such that it is in a position where the velocity fluctuations are greatest, a periodic change in the resistance of the wire will occur of the same frequency as the frequency of formation of the vortices, and the resulting variations of P.D. across the wire, after being amplified, produce a fluctuating anode current superimposed upon the steady anode current in the plate circuit of the last valve. These fluctuations of anode current simultaneously produce induced currents in the crystal circuit of the same periodicity, and, after rectification by the crystal, produce a steady deflexion of the galvanometer, the sensitivity of which is controlled by the shunted resistance S.

The wire A was therefore kept stationary in a most favourable position in one vortex row giving maximum

steady deflexion of the galvanometer, while the other wire, B, was moved along a line up and down the stream in the opposite vortex row, which also gave a maximum steady deflexion of the galvanometer of the same magnitude as that produced by the wire A. The heating currents for both wires were kept constant for all positions of the wires, and the resulting deflexion of the galvanometer due to the periodic heating and cooling of both wires acting simultaneously was observed. When the two wires were in such positions that the velocity fluctuations at these points were in phase, the fluctuating P.D.s across the wires being also

TABLE I.

Amplifier Results for Cylinders.

$D = 4.20$ cm. $N = 150$. $V = 317.5$ cm./sec.

$$\frac{V}{ND} = 5.05 \quad \frac{VD}{v} = 900.$$

Distance behind cylinder x , cm.	$\frac{x}{D}$	h , cm.	l , cm.	$\frac{h}{l}$	$b = \frac{l}{D}$	$a = \frac{u}{V}$	$\frac{b}{1-a} = \frac{V}{ND}$
1.00	2.38	.375	1.375	.274	3.26	.353	5.03
2.20	5.24	.425	1.625	.263	3.86	.230	5.01
3.00	7.14	.450	1.675	.270	3.97	.212	5.06
3.85	9.17	.475	1.70	.284	4.05	.195	5.02
5.00	11.90	.500	1.75	.286	4.17	.170	5.02
5.65	13.45	.502	1.75	.287	4.17	.170	5.02

in phase after amplification produced a reinforcement of the fluctuating anode current, with a consequent maximum steady deflexion of the galvanometer after rectification. When the two wires were in relative positions corresponding to a phase difference of 180° , the fluctuating P.D.s across them opposed each other, resulting in a minimum rectification, and hence a minimum steady deflexion of the galvanometer.

Hence to obtain the longitudinal spacing of the vortices, the galvanometer deflexions due to the rectified currents were observed for corresponding positions of the movable hot wire downstream, and by plotting these deflexions against the distance of the movable hot wire behind the obstacle downstream, similar results were obtained to those

in the previous method, the graph exhibiting maximum and minimum values of deflexion.

The distance between two successive maxima or minima was assumed a measure of the longitudinal spacing between successive vortices in the same row.

TABLE II.

$$D = .650 \text{ cm.} \quad N = 121. \quad V = 437 \text{ cm./sec.}$$

$$\frac{V}{ND} = 5.57. \quad \frac{VD}{\nu} = 1925.$$

Distance behind cylinder x , cm.	$\frac{x}{D}$	h , cm.	l , cm.	$\frac{h}{l}$	$b = \frac{l}{D}$	$a = \frac{u}{V}$	$\frac{b}{1-a} = \frac{V}{ND}$
1.80	2.76	.580	2.325	.250	3.57	.358	5.57
2.50	3.85	.650	2.50	.260	3.85	.310	5.59
4.50	6.91	.730	2.675	.276	4.11	.260	5.56
6.50	10.00	.820	2.80	.293	4.30	.170	5.55

TABLE III.

$$D = .32 \text{ cm.} \quad N = 186. \quad V = 298 \text{ cm./sec.}$$

$$\frac{V}{ND} = 5.01. \quad \frac{VD}{\nu} = 645.$$

Distance behind cylinder x , cm.	$\frac{x}{D}$	h , cm.	l , cm.	$\frac{h}{l}$	$b = \frac{l}{D}$	$a = \frac{u}{V}$	$\frac{b}{1-a} = \frac{V}{ND}$
.75	2.34	.300	1.00	.300	3.13	.376	5.01
1.25	3.90	.350	1.275	.275	3.97	.210	5.02
1.90	5.93	.405	1.275	.317	3.97	.210	5.02
2.50	7.81	.420	1.30	.324	3.90	.223	5.00
4.50	14.10	.460	1.320	.349	4.12	.178	5.00

Results for cylinders are included in Tables I., II., III., and IV. and figs. 2, 3, and 4. It will be observed that each graph is characteristic of a decrease in the maximum value of the peaks, together with an increase in the longitudinal spacings for increasing distances behind the obstacle, evidently due to a widening out of the vortices, resulting in

a decrease in vortex strength as they move down the stream. Such an effect is consistent with the results observed by Fage and Johansen * for a large plate, using a different method.

TABLE IV.

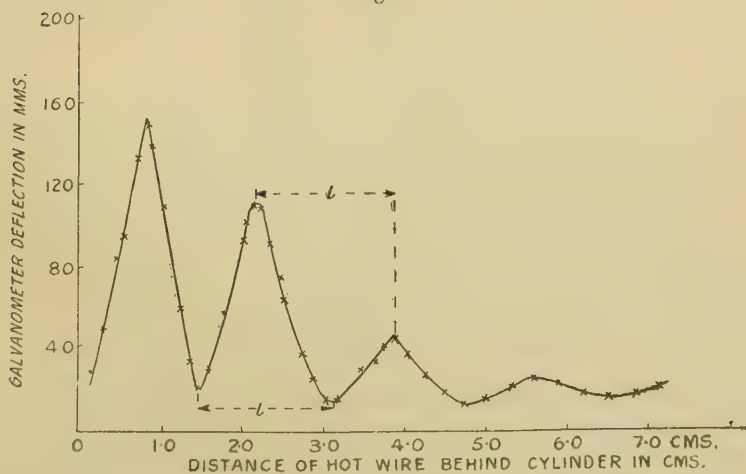
Results by Vibration Galvanometer Method.

$D = 3.58$ cm. $N = 10$ cm. $V = 202.7$ cm./sec.

$$\frac{V}{ND} = 5.66. \quad \frac{VD}{v} = 4900.$$

Distance behind cylinder x , cm.	$\frac{x}{D}$	h , cm.	l , cm.	$\frac{h}{l}$	$b = \frac{l}{D}$	$a = \frac{u}{V}$	$\frac{b}{1-a} = \frac{V}{ND}$
11.50	3.20	3.80	13.00	.292	3.64	.358	5.67
16.50	4.60	4.25	13.50	.314	3.77	.334	5.66
25.00	6.99	4.80	14.00	.342	3.90	.308	5.64
31.50	8.80	5.00	14.75	.339	4.12	.272	5.66

Fig. 2.



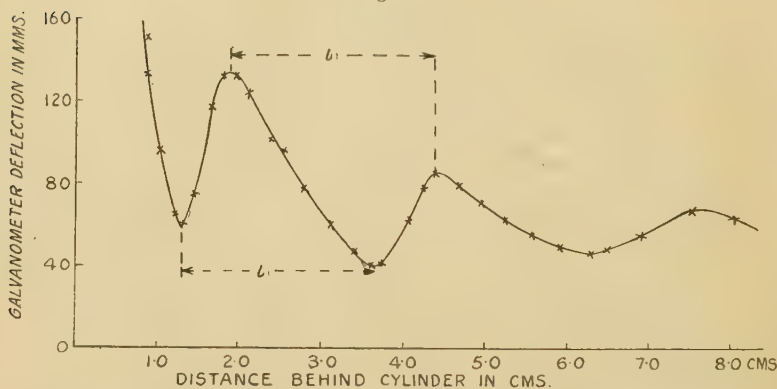
Cylinder $D = .42$ cm.; $V = 317.5$ cm./sec.; $N = 150$; $\frac{V}{ND} = 5.05$.

By using a loud speaker in the plate circuit of the last valve, the positions having the same phase and also 180° out

* Proc. Roy. Soc. A, cxvi. (1927).

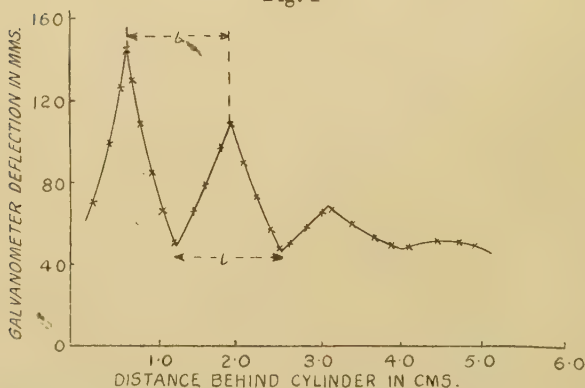
of phase, corresponding to maximum and minimum peaks on the curves, could easily be demonstrated by the audible note heard in the loud speaker ; for since reinforcement of the

Fig. 3.



Cylinder $D = .65$ cm. ; $V = 437$ cm./sec. ; $N = 121$; $\frac{V}{ND} = 5.57$.

Fig. 4



Cylinder $D = .32$ cm. ; $V = 298$ cm./sec. ; $N = 186$; $\frac{V}{ND} = 5.01$.

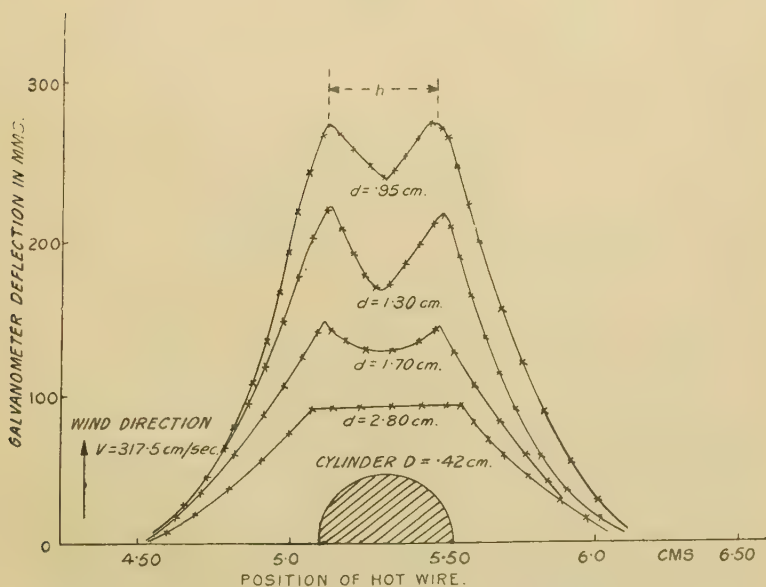
fluctuating P.D.'s across the wire occurs when the wires are in relative positions of same phase, maximum intensity of sound is heard in the loud speaker, whereas, when in positions corresponding to 180° out of phase, minimum resultant P.D. due to both wires produces minimum intensity of sound in the loud speaker. Thus, when the movable hot wire is traversed downstream, alternately maximum and minimum sound is

heard. As the distance of the movable hot wire downstream increased, the maximum intensity showed a gradual falling off, and at a distance of 8 cm. no sound was heard at all, again indicating a diminution in vortex strength.

(2) Determination of the Lateral Spacing of Vortices behind Cylinders in Air.

By using one vertical hot wire transformer-coupled to the amplifier as in fig. 1, it was traversed laterally in a vertical plane at right angles to the wind stream at different distances behind the obstacle.

Fig. 5.



d = distance of lateral traverse behind cylinder.

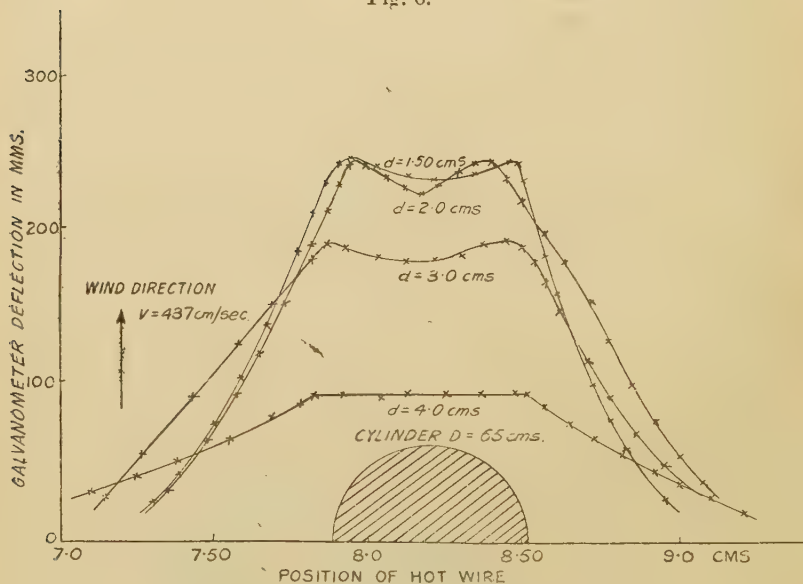
At each position of the wire, the heating current was kept constant, and the corresponding steady deflexions of the galvanometer observed. Figs. 5 and 6 are typical results for cylinders of $D = 0.65$ cm. and 0.42 cm.

It will be observed that each traverse produces a curve exhibiting two maxima, similar to the curves obtained by a vibration galvanometer method described elsewhere*

* *Loc. cit.* ref. †, p. 1113. See also Table IV. for results.

and are similar to Dryden and Kuethe's* results using a different arrangement. The distance between such maxima was again taken as a measure of the lateral distance (h) between the centres of two rows of vortices, for this is true, assuming the maximum variation in velocity amplitude occurs along the paths of the centres of the vortices, which would therefore produce a maximum effect on the periodic heating and cooling of the wire, and hence a maximum deflexion of the galvanometer due to maximum rectification.

Fig. 6.



a = distance of lateral traverse behind cylinder.

The peaks of figs. 5 and 6 also show that as the distance of the vortices behind the cylinders is increased there is a diminution in their peak values, indicating a decrease in vortex strength. At a distance of 4.0 cm., overlapping of the two rows of vortices is sufficient to produce a region of uniform velocity variation, for the traverses show no peaks, but are merely flat over this region.

Fig. 7 shows the variation of maximum velocity amplitude with increase in distance downstream behind a cylinder

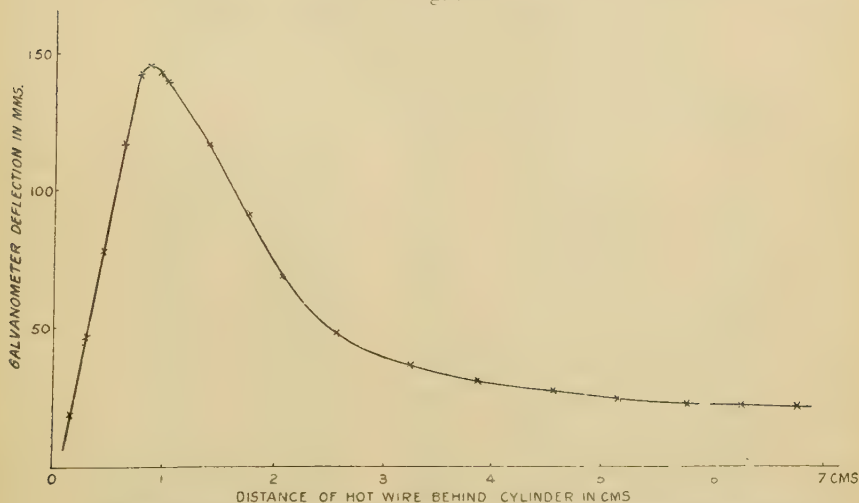
* Report N. A. C. No. 329, Aeronautics, Bureau of Standards, Jan. 8th, 1929.

of $D = .65$ cm. From this figure it will be seen that at first the amplitude is small, but grows quickly, and reaches a maximum value at a distance of 1.0 cm. behind it. It then falls off gradually.

(3) *Frequency Determination of Vortices in Air behind Cylinders, Inclined Plates, and Aerofoils.*

The same arrangement as used in the previous section for the lateral spacings was now employed to determine the

Fig. 7



Variation of maximum velocity amplitude behind cylinder $D = .65$ cm. Traverse downstream at .325 cm. from centre of wake, $V = .445$ cm./sec ; $N = 138$.

frequency of formation of vortices, and in principle is similar to the method first successfully used by Piercy and Richardson *, except that greater amplification was now produced.

By suitably placing the hot wire in one vortex row, where maximum variation in velocity amplitude occurs, and inserting in the plate circuit of the last valve either a pair of telephones or a loud-speaker, the small oscillating P.D.s produced across the wire by its periodic heating and cooling due to formation of vortices, after amplification were made to produce an audible note in either the phones or the loud speaker, of the same frequency as the vortex formation.

* Phil. Mag. vi. Nov. 1928.

With such amplification the note heard was of sufficient intensity to be heard all over a large room, and demonstration to an audience could easily be made.

The value of this frequency was determined by comparing the note heard with a similar note produced on a monochord standardized with a 256 fork. Tables V., VI., and VII. incorporate results for cylinders, inclined plates, and aerofoils, and fig. 8 shows a comparison of results for plates and aerofoils.

TABLE V.
Amplifier Results for Cylinders in Air.
 ν for air = .148.

Diameter D, cm.	Frequency N, per sec.	Velocity V, cm./sec.	$\frac{V}{ND}$	$\frac{ND}{\nu}$
.61	86.5	268	5.10	1100
.61	57.7	182	5.18	750
.42	139	293	5.02	830
.42	110.7	238	5.12	677
.31	181	297	5.30	621
.31	100	165	5.32	345
.163	300	266	5.42	292
.163	360	327	5.58	360
.120	510	330	5.40	268
.120	265	185	5.80	150
.060	335	152.5	7.60	61.8
.060	730	278	6.35	113
.060	820	307.5	6.25	125
.046	373	150	8.75	46.8
.046	453	167	8.00	52.0
.046	645	208	7.02	65

It will be observed that the results are in good agreement with those obtained by other methods*. In particular it is of interest to compare the cylinder results with those of Piercy and Richardson, who obtained a few results for thin rods at somewhat higher values of $\frac{VD}{\nu}$. Good agreement is shown for similar values of $\frac{VD}{\nu}$; but, furthermore, the results

* Relf and Simmons, *Phil. Mag.* xlix. p. 509 (1925); Richardson, *Proc. Phys. Soc.* xxxvii. p. 178 (1925).

in Table V. indicate that at low values of $\frac{VD}{\nu}$ for the cylinders there is an increase in $\frac{V}{ND}$ which is consistent with results obtained by other experimenters*.

TABLE VI.
Amplifier Results for Plates in Air.

Type of Plate.	Angle of Inclination, θ° .	Frequency N, per sec.	Velocity V, cm./sec.	$\frac{Nb}{V}$.	$\frac{Nb \cdot \sin \theta}{V}$.
Length = 11 cm. Width b , = .425 cm. Thickness D, = .50 mm.	5	695	371	.783	.068
	10	422.5	265	.678	.118
	10	490	318	.656	.114
	15	299	287.5	.442	.114
	30	293	425.5	.292	.146
	45	214	425.5	.213	.150
	60	160.7	417.5	.163	.141
	90	152	450	.144	.144
Length = 11 cm. Width b , = .59 cm. Thickness D, = .50 mm.	0	656	384	1.01	0
	0	898	485	1.09	0
	0	579	375	.910	.079
	10	415	371	.660	.115
	17	372	480	.456	.134
	30	237	480	.292	.146
	45	160.5	465	.204	.144
	70	124.5	465	.158	.148
	90	116	465	.147	.147

THEORY AND DISCUSSION OF RESULTS.

Longitudinal and Lateral Spacing of Vortices behind Cylinders.

Referring to Tables I., II., III., and IV., it will be observed that the mean value of $\frac{h}{l} = .288$ agrees fairly well with Karman's theoretical stability condition for two parallel

* Benard, *Comptes Rendus*, cxlvii. p. 839 (1908); clxxxii. No. 25, June 21, 1926. Walton, *Scientific Proc. Roy. Dublin Soc.* xviii. No. 47, Jan. 9, 1928.

rows of vortices, viz., $\frac{h}{l} = \cdot 283$, although the variations are too large to be experimental errors. The widening out of the vortex rows is associated with a corresponding increase in the longitudinal spacings (l), and both these effects produce a retardation in the relative speed (u) of the vortices with

TABLE VII.

Amplifier Results for Aerofoils in Air*.

Type of Aerofoil.	Angle of Inclination, θ° .	Frequency N , per sec.	Velocity V , cm./sec.	$\frac{Nb}{V}$.	$\frac{V}{ND}$.	$\frac{Nb \cdot \sin \theta}{V}$.
Width b , = 1.2 cm. Maximum thickness D , = 1.88 mm.	0	264	278	1.13	5.62	0
	0	396	402.5	1.18	5.40	0
	5	377	445	1.02	6.30	.089
	10	290	358	.970	6.58	.104
	10	280	363	.928	6.89	.099
	15	230	363	.756	8.40	.195
	20	170	337.5	.600	10.55	.205
	25	156	390	.479	13.30	.202
	30	120	425	.339	18.80	.169
	45	100	571	.210	30.40	.148
Width b , = .64 cm. Maximum thickness D , = .82 mm.	0	758	475	1.02	7.66	0
	0	660	402	1.05	7.45	0
	5	661	435	.971	8.03	.089
	5	565	372	.974	8.06	.085
	15	352	382	.590	13.20	.152
	20	300	347	.553	14.18	.189
	20	390	450	.555	14.10	.190
	30	250	450	.355	22.00	.177
	45	145	450	.205	37.90	.145
	60	90	347	.166	47.00	.145

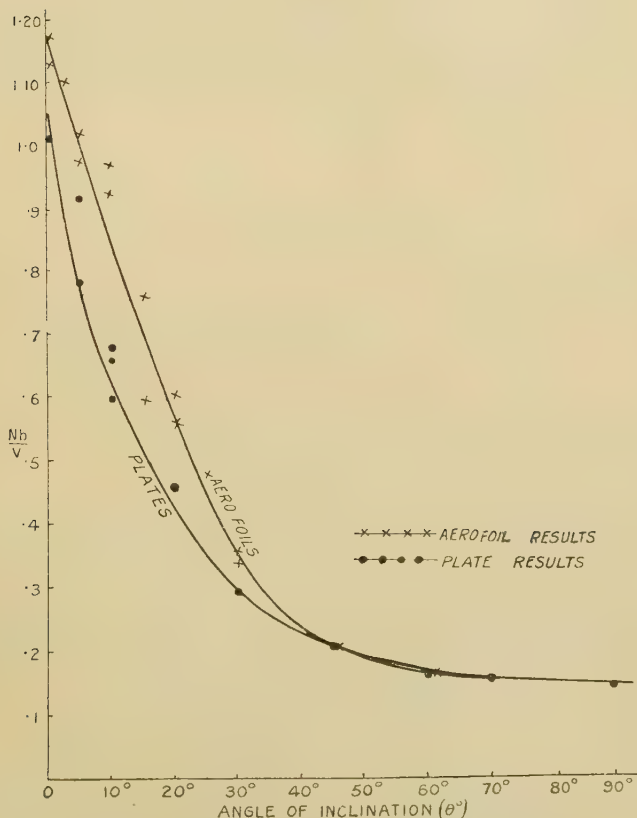
respect to the fluid stream as they recede from the cylinder, for there is a marked decrease in $\frac{u}{V}$ as the distance downstream (x) is increased.

* Phil. Mag. v. March 1928. The following errata in that paper should be noted:—P. 453, Table III. Values of N given are only half correct values for aerofoil No. 1 (Brass). The corresponding values of $\frac{V}{ND}$ should read half the values shown.

By plotting $\frac{l}{D}$ against $\frac{D}{x}$ for the different cylinders, as in fig. 9, the results are represented fairly well by the linear relation

$$\frac{l}{D} = 4.50 \left(1 - 0.62 \frac{D}{x} \right), \dots \dots \dots (1)$$

Fig. 8.



Amplifier results for inclined plates and aerofoils.

which compares favourably with Karman and Rubach's* value of $\frac{l}{D} = 4.30$ for a cylinder in water and Fage's,† value $\frac{l}{D} = 4.27$ for a cylinder in air.

* *Phys. Zeitschr.* xiii, p. 49 (1912).

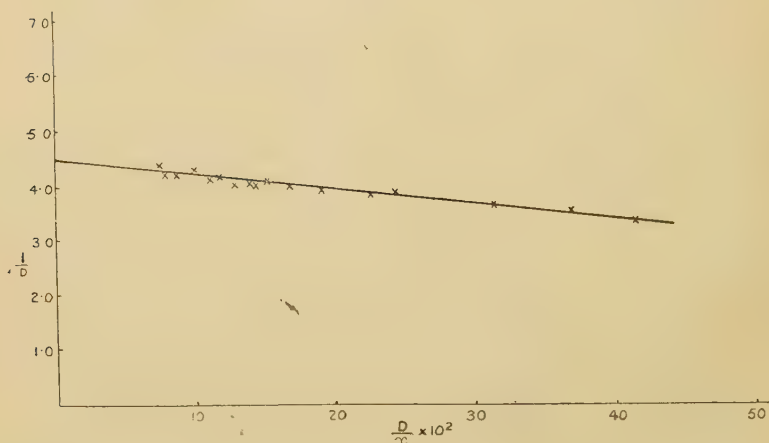
† Roy. Aero. Soc. Jan. 1929.

Beyond $\frac{x}{D} = 12$, the variations of $\frac{l}{D}$ are of the same order as the experimental errors, and it is only at such distances behind the cylinders that the speed of the vortices relative to the surrounding fluid might be assumed constant.

Frequency Formation of Vortices. Relation between $\frac{ND}{V}$ and $\frac{VD}{\nu}$ for Cylinders in Air.

Following the method adopted by Rayleigh* and others, by applying the theory of dimensional similarity, the

fig. 9.



frequency of formation of the vortices, N , formed behind a stationary cylinder of diameter D cm. can be expressed as

$$\frac{ND}{V} = K f\left(\frac{\nu}{VD}\right),$$

where V = velocity of the fluid stream,
 ν = kinematic viscosity of the fluid,
 K = a constant,

and the function $f\left(\frac{\nu}{VD}\right)$ is unknown, and is to be determined from the experimental results.

Assuming that $\frac{ND}{V}$ would be affected by a change in

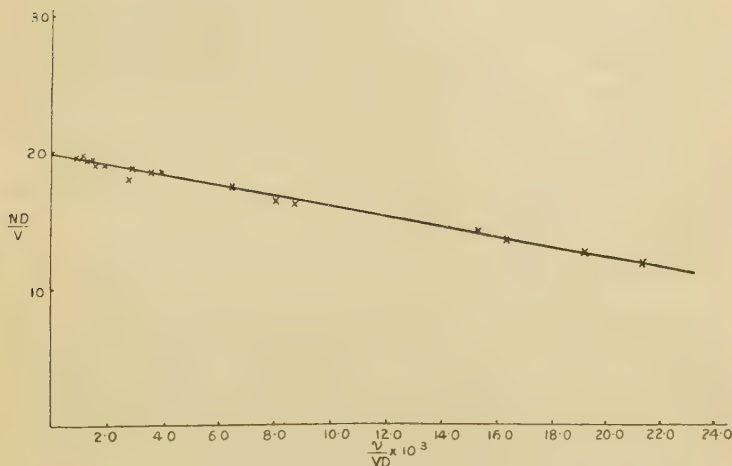
* Rayleigh's 'Sound,' ii. pp. 413, 414.

viscosity ν , and expanding $f\left(\frac{\nu}{VD}\right)$ as a Maclaurin series, we get

$$\frac{ND}{V} = a + b \cdot \left(\frac{\nu}{VD}\right) + c \left(\frac{\nu}{VD}\right)^2 + \dots \text{etc.},$$

where a , b , and c are constants.

Fig. 10.



Since generally $\frac{\nu}{VD}$ is small with respect to $\frac{ND}{V}$, neglecting the higher power terms above the first order, we have

$$\begin{aligned} \frac{ND}{V} &= a + b \cdot \frac{\nu}{VD} \\ &= A_1 \left(1 + B_1 \frac{\nu}{VD}\right), \quad \dots \dots \dots (2) \end{aligned}$$

where A_1 and B_1 are also constants.

This is a linear equation in $\frac{ND}{V}$ and $\frac{\nu}{VD}$, and by plotting the values of these non-dimensional quantities from Table V. a straight-line graph, as in fig. 10, is obtained, from which values of $A_1 = 19.8$ and $B_1 = -19.7$ are deduced.

Substituting these values in equation (2), the experimental results can be well represented by the equation

$$\frac{ND}{V} = 19.8 \left(1 - 19.7 \cdot \frac{\nu}{VD}\right), \quad \dots \dots (3)$$

which is nearly identical with that deduced by Rayleigh from Strouhal's* results obtained by a different method, A_1 and B_1 having values of $\cdot 195$ and $20\cdot 1$ respectively.

Results for Inclined Plates and Aerofoils.

Analysis of the results for inclined plates suggests that at angles of inclination (θ) between $20^\circ - 90^\circ$, $\frac{Nb \cdot \sin \theta}{V} = \cdot 148$, and is constant for different plates, whereas the aerofoils show this constancy at angles of 30° upwards. This is only true when the effect of viscosity is small, as will be shown later.

Between 0° and 30° there is, however, a difference between the plate and aerofoil results (see fig. 8) which is to be expected, since at low angles of inclination the maximum thickness D of the aerofoil plays a more important part in controlling the rate of formation of the vortices, and $\frac{ND}{V}$ must now replace $\frac{Nb \cdot \sin \theta}{V}$. Variations of $\frac{V}{ND}$ with θ are included in Table VII. for the aerofoils.

A comparison of the results with those of Fage and Johansen† shows excellent agreement for 30° upwards. It is therefore highly probable that the vortex frequency formation behind an inclined plate at θ° to the fluid stream, and of width b cm., is similar to that behind a plate normal to the stream and having effective width $b \cdot \sin \theta$.

In such a case, both the lateral and longitudinal spacings will be modified proportionally. Confirmation of this has been made by Fage and Johansen, who found that $\frac{l}{b \cdot \sin \theta} = 5\cdot 32$, and is constant between angles $30^\circ - 90^\circ$. Some later work‡, by a different method, has also verified this constancy of $\frac{l}{b \cdot \sin \theta}$.

Effect of Viscosity.

By similar dimensional treatment to that used for the cylinders it can be shown by replacing the diameter D of the cylinder by $b \cdot \sin \theta$, the effective width of the plate, and neglecting the power terms of viscosity, that

$$\frac{Nb \cdot \sin \theta}{V} = A_2 \left(1 + B_2 \frac{\nu}{V \cdot b \cdot \sin \theta} \right),$$

* *Ann. der Phys.* v. p. 216 (1878).

† *Loc. cit.* ref. *, p. 1117.

‡ To be published later.

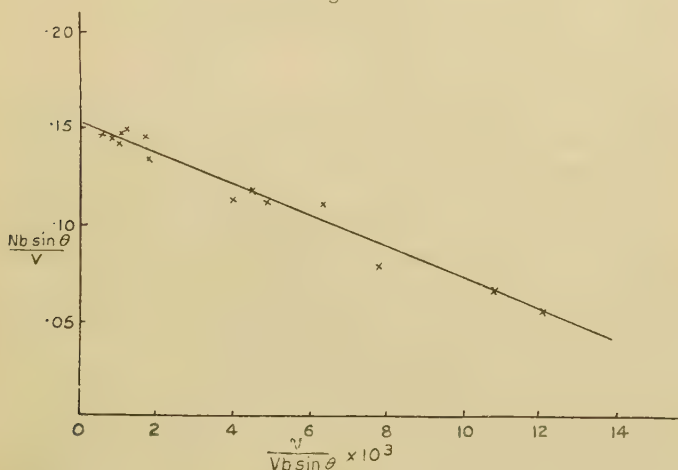
and a plot of $\frac{Nb \cdot \sin \theta}{V}$ against $\frac{v}{Vb \cdot \sin \theta}$ also gives a linear relation. Fig. 11 shows this quite well, the constants A_2 , B_2 having values $\cdot 152$ and $-52\cdot 6$ respectively. Thus, retaining the viscosity term, the equation

$$\frac{Nb \cdot \sin \theta}{V} = \cdot 152 \left(1 - 52\cdot 6 \frac{v}{Vb \cdot \sin \theta} \right)$$

is representative of the plate results over the whole range investigated.

It is of interest to compare these results with those of Karman and Rubach * for a plate normal to a steady stream

Fig. 11.



of water, for without applying a correction for viscosity they obtained a value of $\cdot 145$ for $\frac{N \cdot b}{V}$.

Further investigations are still in progress, the results of which will appear in a later paper.

SUMMARY.

A differential method, using two electrically heated platinum wires in conjunction with a valve amplifier is described for measuring the longitudinal spacing (l) of vortices formed behind cylinders of small diameter placed in a steady air-stream. Measurements of the lateral

* *Loc. cit.* ref. *, p. 1125.

spacings (h) of the vortex rows are also included, together with values of $\frac{h}{l}$, and a comparison with theory shows fairly good agreement for the average results only. A determination of the frequency of formation of the vortices behind cylinders, inclined plates, and aerofoils is made by using the same amplifier, and empirical formulæ included to account for the results.

Physics Dept.,
College of Technology,
Leicester.
January 1st, 1930.

CIII. *On the Mathematical Representation of Sensibility to Difference of Colour.* By R. A. HOUSTOUN, D.Sc.,
Lecturer on Physical Optics in the University of Glasgow *.

IT is well known that a colour possesses three qualities, hue, brightness, and saturation, and that it can consequently be specified by three independent variables. Also any light can be matched in colour and brightness by a superposition of three colours, the so-called primary colours. These may be real, if negative quantities are permitted in the mixture, but are otherwise imaginary. As independent variables we may choose the quantities of the primaries occurring in the mixture, measured in energy units. This was the method adopted by Helmholtz; thus according to Helmholtz the three variables are of exactly the same nature. According to Hering, on the other hand, the colour possesses brightness and two colour valencies, and two of the variables are of a different nature from the third.

The merits of these rival standpoints have been a subject of debate for years. Helmholtz's view had the advantage of a definite physical model, three different systems of nerves corresponding to his three primary colours, which made a stronger appeal to the imagination than Hering's somewhat vague physiological processes.

* Communicated by the Author.

As the result of a paper already published* and of some experimental work not yet published, I have found that the sensation of brightness is strictly proportional to the number of nerves excited. Colour may therefore be due to the number of impulses passing along each nerve per second. This gives a new standpoint on the subject, and there is also other recent work the bearing of which has not yet been investigated. I have therefore thought it worth while to re-examine the question.

The first requirement which a theory of colour-vision has to satisfy is the laws of colour mixing. The Young-Helmholtz theory does this in the most satisfactory manner on account of the definite picture which it gives of the process, but any other theory the variables of which are linear functions of the Young-Helmholtz variables gives the same numerical results as the latter does. This was pointed out specifically by Helmholtz † for the case of Hering's theory, and I have illustrated the point by some calculations ‡.

The next requirement to be satisfied is the perception of difference of colour. This point has not received so much attention in the discussion of the subject as the laws of colour mixing. The problem may be stated as follows: Let the three Young-Helmholtz variables, *i. e.*, the magnitudes of the stimuli affecting the three nerve systems, be laid off parallel to rectangular coordinate axes. Then to every colour there corresponds a point in space. How far must we move from any given point for the colour to become appreciably different? Or, putting it mathematically, x_1, x_2, x_3 is given, and the quantities of the variables in the mixture are altered until there is a just appreciable change in sensation; what condition must dx_1, dx_2, dx_3 satisfy?

Let dS be the change in the resultant sensation while dS_1, dS_2 , and dS_3 are the changes in the three fundamental sensations. Then Helmholtz assumed

$$\begin{aligned} dS^2 &= dS_1^2 + dS_2^2 + dS_3^2 \\ &= \frac{1}{3} \left(\frac{dx_1^2}{x_1^2} + \frac{dx_2^2}{x_2^2} + \frac{dx_3^2}{x_3^2} \right). \end{aligned}$$

* Phil. Mag. viii. p. 520 (1929).

† *Physiologische Optik*, Auflage ii., p. 376.

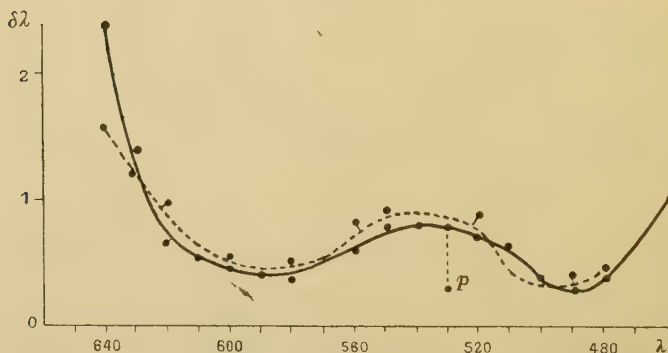
‡ Phil. Mag. xxxviii. p. 402 (1919).

This expression had the advantage that when x_2 and x_3 were zero it reduced to

$$\alpha S = \frac{dx_1}{\sqrt{3x_1}},$$

which agreed with Fechner's law. Helmholtz tested the expression by applying it to König's* measurements on the discrimination of hue in the spectrum. The results are shown in fig. 1, which is taken from Helmholtz's 'Physiological Optics,' fig. 1, 2nd Edition, p. 455. The

Fig. 1.



full line gives the difference of wave-length which according to König could just be discriminated at different points in the spectrum. The broken line shows what it should be according to the above assumption. König and Dieterici † had already determined S_1, S_2, S_3 as functions of λ ; Helmholtz found that their primaries gave no agreement, and from their results calculated other primaries which gave the result shown.

With reference to the agreement it may be stated that the curve in the diagram shows only two points of maximum sensibility to change of wave-length in the spectrum, one in the yellow and one in the blue-green, whereas it is now known there are four such maxima. Also E. Schrödinger ‡ has pointed out that Helmholtz's

* König and Dieterici, *Ann. d. Phys.* xxii. p. 579 (1884).

† König and Dieterici, *Berl. Ber.* p. 805 (1889).

‡ *Ann. d. Phys.* lxxiii. p. 481 (1920).

expression leads to a quite impossible value for the brightness of the solar spectrum, "a horrible camel's back with two well-marked maxima." Moreover, it has been shown by Abney and others that the brightness of colours is additive. This result, which is now appropriately known as Abney's law, is quite incompatible with Helmholtz's assumption.

Abney experimented with a spectrum falling upon a screen in which slits could be opened, so as to transmit different colours separately and together. These were allowed to illuminate a surface, and the intensity of the illumination falling on this surface was measured photometrically against white light from the same source. The red was thus, for example, equal to so many metre-candles of white light, the green to so many metre-candles, and the red and green together to a number of metre-candles equal to the sum of the separate values. Thus in computing the brightness of an illumination we can neglect the colour of the components.

Helmholtz's assumption is therefore out of the question, and Schrödinger has sought to replace it by

$$dS^2 = \frac{1}{a_1x_1 + a_2x_2 + a_3x_3} \left(\frac{a_1dx_1^2}{x_1} + \frac{a_2dx_2^2}{x_2} + \frac{a_3dx_3^2}{x_3} \right),$$

which is in agreement with Abney's law. He tests his expression by the experimental results shown in fig. 1, and finds that the agreement is about as good as that obtained by Helmholtz. When x_2, x_3, dx_2, dx_3 are equal to zero his expression reduces to

$$dS = \frac{dx_1}{x_1},$$

which agrees with Fechner's law.

The fact that both Helmholtz's and Schrödinger's assumptions reduce to the conventional form for Fechner's law shows that they cannot hold for more than a limited range of intensities, for it is only over a limited range that this law represents the results even approximately.

Let us assume we are dealing with the colour for which

$$x_3=0 \text{ and } x_1=nx_2=\frac{n}{n+1} x \text{ say. Then}$$

$$dx_1=n dx_2=\frac{n}{n+1} dx,$$

and according to Helmholtz

$$dS^2 = \frac{1}{3} \left(\frac{dx_1^2}{x_1^2} + \frac{dx_2^2}{x_2^2} \right) = \frac{2}{3} \left(\frac{dx}{x} \right)^2,$$

whence

$$dS = \sqrt{\frac{2}{3}} \frac{dx}{x}.$$

According to Schrödinger

$$\begin{aligned} dS^2 &= \frac{1}{a_1x_1 + a_2x_2} \left(\frac{a_1dx_1^2}{x_1} + \frac{a_2dx_2^2}{x_2} \right) \\ &= \frac{n+1}{a_1nx + a_2x} \left(\frac{a_1ndx^2}{(n+1)x} + \frac{a_2dx^2}{(n+1)x} \right) \\ &= \left(\frac{dx}{x} \right)^2, \end{aligned}$$

and

$$dS = \frac{dx}{x}.$$

Thus for a composite colour Helmholtz's assumption does not give the same expression for Fechner's law as it does for a primary colour, while Schrödinger's expression does. Now König and Brodhun * tested this point specially, and found that the law was the same, no matter whether the colour was primary or composite. Thus on this point Schrödinger's expression is decidedly superior.

The criticism to which Schrödinger's expression might be subjected is that he abandons the standpoint of independent sensations. The three variables defining the stimulus, x_1 , x_2 , x_3 , are independent, but the increment dS corresponding to a given dx depends not only on x_1 , but also on x_2 and x_3 . The extent to which the red sensation is affected by a given change of stimulus depends on whether the green sensation is simultaneously stimulated or not. Thus the hypothesis of three independent sensations is fitted to Fechner's law by abandoning their independence. I feel certain that the adjustment cannot be made in any other way.

For let x_1 be the energy absorbed per second in stimulating the one nerve system, x_2 the energy absorbed per second in stimulating the other, and let S_1 , S_2 be the two

* *Sitz. k. Akad. Wissensch.* (Berlin) p. 917 (1888).

sensations. Then by choosing different units for x_1 and x_2 we may write

$$S_1 = f(x_1), \quad S_2 = f(x_2).$$

The total sensation is given by

$$S = f(x_1) + f(x_2).$$

But, experimentally,

$$S = f(ax_1 + bx_2),$$

where a and b are constants, since it is necessary to express the energy absorbed in a third unit. This requires that S should be proportional to x , which is impossible.

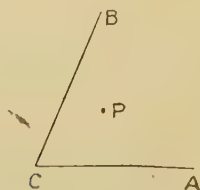
The expressions which Helmholtz and Schrödinger postulate for dS^2 represent both change of brightness and change of hue and saturation. Let us suppose that in a colour matching apparatus the two fields are a perfect match as regards brightness, hue, and saturation, and let us increase the brightness of the one field. Then keeping the brightness constant let us change the colour. The eye recognizes the changes as different in kind. Consequently it seems wrong to combine them in the same expression.

I think it desirable that one of the three variables defining the sensation should represent the brightness and nothing but the brightness. This satisfies Abney's law and Fechner's law automatically. It is an unusual standpoint for a physicist to adopt, but, as has been mentioned, I have obtained the result elsewhere that the brightness is proportional to the number of percipient elements active in the retina; thus Abney's law is satisfied, because there is always the same number of percipient elements on both sides of the equation. The mathematician will observe that my assumption makes it easier for the other two variables; they have now only to fulfil the requirements of colour mixing and colour sensitivity, and have nothing whatever to do with Abney's law or Fechner's law.

Let us now consider the nature of the other two variables. They have in the first place to satisfy the laws of colour mixing. These laws are summed up in the centre of gravity construction which was introduced in a sketchy manner in Newton's 'Optiks' and made into an accurate instrument of calculation by Maxwell and Helmholtz.

Let us assume that we have any three coloured lights, and that we superimpose α units of the first, β of the second, and γ of the third. Then if we admit negative values of α , β , and γ , we can produce in this way all possible colours. Place α , β , and γ at the points A, B, and C (fig. 2), regard them as masses, and let P be their centre of gravity. The position of P depends obviously only on the ratios of α , β , and γ , not on their absolute magnitudes, *i.e.*, it depends only on the colour of the resultant light, not on its brightness. All colours occurring in nature are represented in the diagram, and it may be shown that, if we take any two colours and superimpose them, the colour of the resultant is obtained by placing masses on the diagram at the points representing the component colours equal in magnitude to the "quantities" of the components and taking the centre of gravity of these masses. It represents

Fig. 2.



the colour of the resultant. The "quantity" of a component is the sum of its α , β , and γ values and is thus usually not proportional to its brightness.

Once the colours are arranged on the diagram, any three points may be regarded as the primaries. The laws of colour mixing in themselves are independent of any particular choice of primaries, and the latter have been chosen from other considerations; for example, in order to make α , β , γ always positive, to satisfy the phenomena of colour blindness, or to be easily produced in the laboratory. They may be real colours or colours not capable of actual production.

Once the colours are arranged on the diagram any two coordinates specifying the position of a point can be taken as the remaining variables of the theory. According to Hering these variables should be the red-green valency and the yellow-blue valency, and according to a recent

paper by Schrödinger * the relations between Hering's valencies and the system of fundamental sensations adopted by König is mathematically both simple and interesting. The question arises as to whether the phenomena of the variation of colour sensibility require one system of coordinates in preference to another. Hering's valencies were chosen to explain the phenomena of simultaneous and successive contrast, but these phenomena are not of such a definite nature as the facts connected with the variation of colour sensibility, and if the latter could be described more easily with reference to Hering's theory, its position would be considerably strengthened.

Whenever the fundamental colours are chosen we are allowed considerable freedom in drawing the centre of gravity construction. We may place B (fig. 2) anywhere on the plane with reference to CA, *i. e.*, we have the two constants defining its position at our disposal. The intensities of the three colours are not necessarily measured in units of equal energy or equal brightness. Thus when we have the unit of α fixed, we have the ratio of the units of β and γ at our disposal. There are thus four constants at our disposal, and there is consequently a fourfold infinity of constructions which will satisfy the laws of colour mixture. Is one of these constructions unique with reference to the phenomena of colour sensibility?

König's measurements on the discrimination of hue in the spectrum, *i. e.*, on the smallest difference in wave-length $\delta\lambda$ which gives rise to an appreciable difference in hue, have already been referred to and represented in fig. 1. He and his collaborator Dieterici each found three maxima of sensibility one of which lies outside the range of the diagram. Their results were obtained by the method of mean error. Their work was repeated by a direct method both by Steindler † in 1906, and by L. A. Jones ‡ in 1917. The latter each found four maxima of sensibility. One set of results obtained by L. A. Jones is represented by the full curve in fig. 3, the ordinate in this case being proportional to $1/\delta\lambda$, so that the maxima represent maxima of sensibility. The results of Steindler and Jones are in good agreement, and there is no doubt whatever about

* *Sitz. d. Wiener Akad.* cxxxiv. (ii. a) p. 470 (1925).

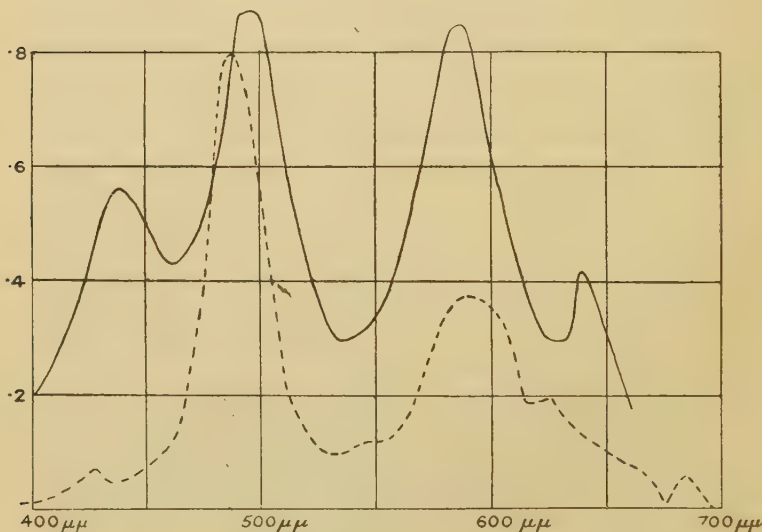
† *Sitz. d. Wiener Akad.* cxv. (ii. a) p. 115 (1906).

‡ *Journ. Op. Soc. of Amer.* p. 63 (1917).

the existence of the fourth maximum in the red ; I got it myself on the first attempt with an improvised apparatus. The sensibility to difference of hue throughout the spectrum does not vary appreciably with the intensity, until the latter becomes very small. It thus forms a very important test for any theory of colour vision. L. A. Jones found altogether 128 distinct hues between $400\ \mu\mu$ and $700\ \mu\mu$.

L. A. Jones and E. M. Lowry * have made a valuable series of determinations of retinal sensibility to saturation

Fig. 3.



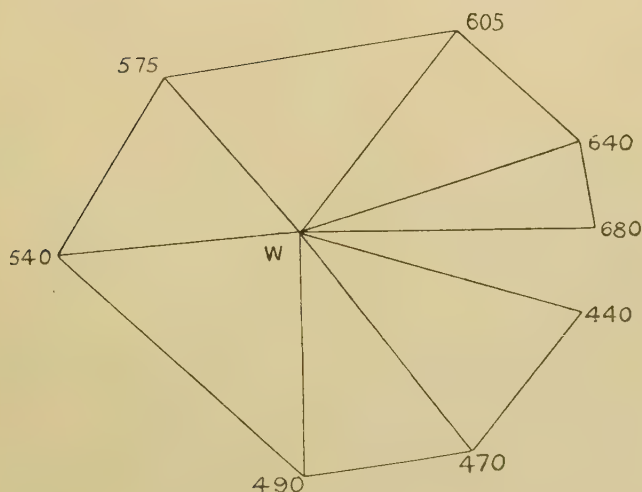
differences. Eight spectrum colours were employed, each was diluted with white, keeping the brightness constant, and the number of just perceptible steps necessary to reach white measured in each case. The apparatus had comparison fields with a sharp line of separation ; one side was diluted with white until there was a just perceptible difference, the other side was then made equal to the first, and a second step taken. Some of the results of Jones and of Jones and Lowry are represented in the following table and in fig. 4.

* Journ. Op. Soc. Amer. xiii, p. 25 (1926).

TABLE.

λ .	Steps to white.	Steps along spectrum.
440 $\mu\mu$	23	14
470	22	13
490	19	26
540	19	16
575	16	23
605	20	13
640	23	7
680	23	

Fig. 4.

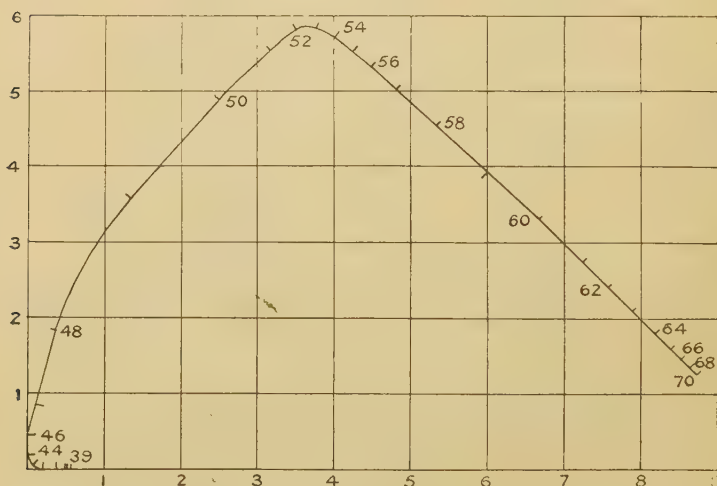


The first column in the table gives the eight wave-lengths employed by Jones and Lowry, the second column the number of steps required to reach white from the wave-length in question, and the third column the number of steps along the spectrum required to reach each wave-length from the next according to the measurements of Jones. In fig. 4 W represents white, and the distances between white and

the different wave-lengths and between consecutive wave-lengths are proportional to the number of steps. Thus fig. 4 represents a rough colour-sensibility diagram. The distance between two colours on it represents the number of just perceptible steps required to reach the one colour from the other.

Fig. 5 represents the König-Ives colour-mixture diagram in rectangular coordinates. It gives the position of the spectrum according to the data of König, as recalculated by Ives; the data of König are regarded at present as

Fig. 5.



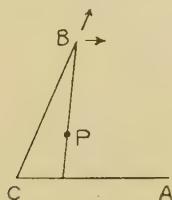
most authoritative, and this diagram is the most convenient way of employing them. Is it possible to modify fig. 5 by changing the four constants at our disposal, so as to make it also serve the same purpose as fig. 4?

Let us consider the effect of changing these four constants. Let P be the centre of gravity (fig. 6) of three masses situated at A , B , and C . Displace B parallel to CB . Then P suffers a displacement parallel to CB proportional to its distance from CA . Displace B parallel to CA . Then P suffers a displacement parallel to CA proportional to its distance from CA . Thus in one case the diagram suffers an extension and in the other a shear. Straight lines remain straight, and the ratio of their parts is unaltered by the displacement.

Let us now diminish the unit in which the mass situated at B is measured. The apparent value of this mass increases, and P moves along PB towards B. Every straight line parallel to CA moves towards B keeping parallel to itself, but obviously the ratio of the parts of the straight line PB is altered by the change. It may be shown by considering the coordinates of three points that straight lines remain straight. Diminishing the unit in which one of the other masses is measured produces similar results.

Thus, no matter how we modify fig. 5, the straight part from the red end to 540 $\mu\mu$ remains straight. Now the corresponding portion of fig. 4 is not straight, and cannot be made straight even if we take a different unit for the least perceptible step in two different directions. If different units of sensibility had been required in the

Fig. 6.



directions corresponding to Hering's colour valencies, that theory would naturally have been strengthened.

Since none of the variants of the colour-mixture diagram can be made to represent sensibility to both changes of hue and of saturation, it is interesting to inquire whether these requirements can be met separately. I have calculated the distances along the curve in fig. 5 which correspond to a difference of 10 $\mu\mu$, and plotted them in the broken curve in fig. 3 against their mean wave-lengths. The result gives the sensibility to difference of wave-length as recorded by the König-Ives diagram. It will be noticed that the curve gives three of the four maxima obtained by L. A. Jones in approximately the correct positions. The fourth one is not given; König's data have not been sufficiently accurate. But the same criticism applies to the evaluations of the colours of the spectrum in terms of the three primaries made by Maxwell, Dieterici, Abney, and Dow and myself*. A redeter-

* *Phil. Mag.* xlv. p. 169 (1923).

mination of the position of the spectrum on the diagram in the light of the desired result would probably reveal a correspondence in this point also.

On examining the König-Ives diagram it will be noticed that the spectrum curve is straight from $690\ \mu\mu$ to $540\ \mu\mu$ and approximately straight from $520\ \mu\mu$ to $440\ \mu\mu$. If we assume these lines as sides of the triangle, it is obvious that by changing the scale of the second side the relative heights of the two principal maxima could be brought into agreement, and that by changing the units in which two of the three primaries are measured the positions of the maxima could be moved along the sides. So that doubtless, apart from the fourth maximum, quite a good agreement could be obtained between the two curves.

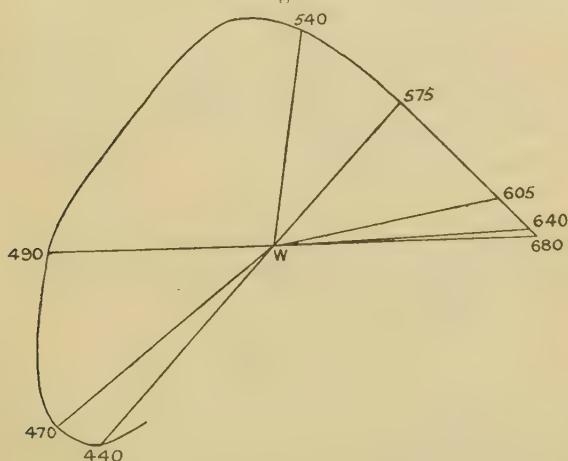
So much for the variation of hue along the spectrum. Let us now consider the sensibility to saturation differences. We have four constants at our disposal in the colour-mixture diagram. It is therefore possible to manipulate the diagram so as to get five lengths on it into a given ratio. According to Jones and Lowry between $680\ \mu\mu$, $575\ \mu\mu$, $540\ \mu\mu$, $490\ \mu\mu$, and $440\ \mu\mu$ and white there are 23, 16, 19, 19, and 23 steps respectively. I resolved therefore to adjust the diagram, so as to make the distances between these points take the ratio of the above numbers. Jones and Lowry employed a white of 5200°K . I found the coordinates of this point to be 0.302, 0.302, 0.396, on the König-Ives diagram. It was selected as origin, and $680\ \mu\mu$ and $575\ \mu\mu$ chosen as the points A and B of fig. 6. An extension of the diagram in the direction CB brought the distances of $680\ \mu\mu$ and $575\ \mu\mu$ into the correct ratio. The next point to adjust was $490\ \mu\mu$. It is very nearly on CA. The unit in which α is measured was altered so as to make the distance of this point right. There then remained two constants at my disposal, the unit in which β was measured and the angle between CA and CB. The remaining two distances were expressed in terms of these constants, and the latter solved for. The details of the calculation are somewhat tedious, so I omit them and content myself with giving the result in fig. 7

This figure is a colour-mixture diagram which has the property, that the distances from white to five of the eight wave-lengths employed by Jones and Lowry are in the ratio of the numbers of steps of saturation difference necessary to reach these wave-lengths. It is found on

examination of the diagram, that the distances to the other three points are not far from the correct ratio. Consequently, the diagram, when drawn in this manner, in addition to representing the colours in terms of the primaries, states approximately their degree of saturation.

Jones and Lowry found that the steps were not equal, being smallest at complete saturation and at white. For example, if we take the case of $680\text{ }\mu\mu$ when it was gradually diluted with white, just perceptible differences occurred at 97.0, 93.9, 90.5, 87.0, etc. per cent. of red, giving differences of 3.0, 3.1, 3.4, 3.5, etc. per cent. The differences gradually increase to a maximum of 6 per cent., and then diminish to 2.8 per cent. on reaching white. Two

Fig. 7.



of the wave-lengths investigated are almost complementaries, $680\text{ }\mu\mu$ and $490\text{ }\mu\mu$. It is consequently possible to follow this direction across the diagram through white to saturation at the other side. The white used is, in units of equal luminosity, 40.5 per cent. of the red, and 59.5 per cent. of the blue at the ends of the path. If we specify the ends of the steps in percentages of this red and blue, the differences start from red at 1.8 per cent., rise to a maximum of 3.5 per cent., diminish to below 2 per cent. in passing through white, then rise to above 3 per cent., and finally diminish to 1.2 per cent. at the blue end. White is thus a point of maximum sensibility. It is impossible by any manipulation of the diagram to make these steps equal.

Let us now collect the results of this part of the investigation. If one of the three sensations is to specify brightness and the other two have to specify colour, it would involve the least disturbance to current practice if the sensations taken were

$$R + G + B, \quad \frac{R'}{R' + G' + B'}, \quad \frac{G'}{R' + G' + B'},$$

where R , G , and B are the Young-Helmholtz fundamental sensations expressed in units of equal luminosity and R' , G' , B' the same sensations in units equal quantities of which make white. But when it comes to specifying sensibility to difference of colour, neither the second and third variables above nor the Hering colour valencies, are suitable. Polar coordinates seem indicated. The origin must be at white, and the scale of the diagram must be adjusted so that the distances to the five wave-lengths of Jones and Lowry are proportional to the number of steps to these wave-lengths. Then a step of sensation outwards will be given by

$$dS_r = f(r) dr,$$

where $f(r)$ is a function which decreases to a minimum near the middle of the range. A step of sensation in a transverse direction will be given by

$$dS_\theta = \frac{F(\theta)}{r} d\theta,$$

where $F(\theta)$ is chosen to bring the dotted curve of fig. 3 into agreement with the full curve. But, from the point of view of getting at the mechanism, at present it seems neither profitable to seek the exact form of these functions nor to investigate the connexion of dS_r and dS_θ with the resultant step.

In the conventional representation of the colour triangle white is placed at the centroid, and the units are then said to be of "equal sensation value." No further definition has been given of what this phrase means. But the procedure has been due to the sound instinct to space difference in colour over the diagram as uniformly as possible. And the shape of the spectrum curve obtained in this way bears a rough resemblance to the curve obtained by the exact procedure of this paper and represented in fig. 7.

CIV. *Energies of Electrons in Gases.* By J. S. TOWNSEND,
M.A., Wykeham Professor of Physics, Oxford*.

1. **T**HE investigations of the electrical properties of gases which have been carried out at the Electrical Laboratory, Oxford, include many experiments connected with phenomena observed with currents of various intensities when the mean energy of agitation of the electrons is of the order of the amount of energy corresponding to a potential of 5 volts.

In the experiments on the determination of the motion of electrons in gases with photo-electric currents it was found that, after the electrons have traversed a certain distance through a gas under the action of an electric force, a steady state of motion is established, where the mean energy of agitation E_1 of the electrons and the mean velocity W_1 in the direction of the electric force are functions of the ratio of the electric force Z to the gas-pressure p .

The energy E_1 and the velocity W_1 were thus obtained experimentally in terms of the ratio Z/p , without making any hypothesis as to the lengths of the mean free path or the amount of energy lost in a collision†.

The mean free path l and the mean loss of energy $\lambda \cdot E_1$ of electrons in collisions with molecules of the gas may be found by the theory of the motion, in terms of the velocities W_1 and the energy E_1 .

The numerical factors that occur in the formulæ from which l and λ are derived have not been determined to a high degree of accuracy, but the most interesting results obtained from the experiments do not depend on the numerical factors, as they give the changes in l and λ obtained with different values of E_1 , and the relative values of these quantities in different gases.

2. In each gas there is a range of forces and pressures where the coefficient λ and the mean free path at unit pressure $L = l \times p$ are approximately constant. These

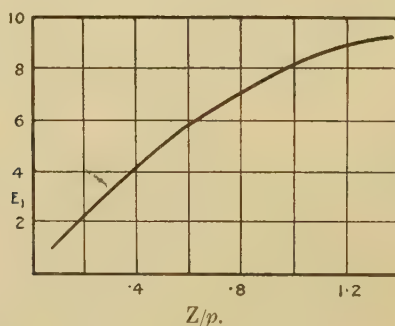
* Communicated by the Author.

† A general description of these experiments and the tables of results obtained with several gases are given in the pamphlet on the 'Motion of Electrons in Gases.' Clarendon Press, Oxford.

ranges are well marked in helium * and in neon †, where the mean energy of agitation E_1 is approximately proportional to the ratio Z/p . At the upper limit to this range in neon, λ is approximately 7×10^{-5} , and the mean energy of agitation of the electrons is about 4 volts. In helium the upper limit to this range is near the point where the mean energy of agitation is 4 volts and λ is approximately 2.5×10^{-4} . With larger energies of agitation the values of λ are considerably increased, and the increases may be attributed to losses of energy in large amounts in some of the collisions where the kinetic energy of the electrons is much greater than the mean energy.

The effect of large losses of energy in neon and in helium may be seen directly from the curves (figs. 1 and 2), which give the mean energy of agitation E_1 in the steady state

Fig. 1.—Neon.



of motion as found experimentally in terms of the ratio Z/p . The ordinates of the curves give the values of E_1 in volts, and the abscissae the ratio Z/p , Z being in volts per centimetre and p in millimetres of mercury.

In each gas there is a considerable portion of the curve which is almost in a straight line, and within this range the changes in L and λ are small. With the larger energies of agitation the rate of increase of E_1 with Z/p diminishes, as is shown by the change of the slope of the curves. This indicates that the proportion of the energy of electrons which is lost in collisions increases with the energy of agitation.

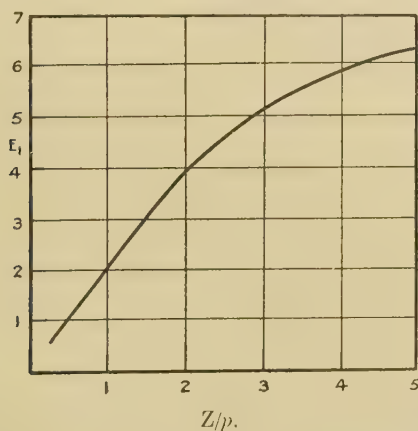
* J. S. Townsend and V. A. Bailey, *Phil. Mag.* xlv. p. 657 (Oct. 1923).

† V. A. Bailey, *Phil. Mag.* xlvii. p. 379 (1924).

The curve (fig. 1), which gives the energy of agitation of electrons in neon, is approximately a straight line ($E_1=10.5 \times Z/p$) from the point where E_1 is 2 volts to $E_1=4$ volts. At the point $Z/p=.8$ the mean energy of agitation is 7.2 volts, which differs considerably from 8.4 volts, to be expected if there were no change in the proportion of the energy of an electron that is lost in a collision. This indicates that the energy of agitation of a considerable proportion of the total number of electrons is much below 7.2 volts.

The energies of agitation of electrons in helium are given by the curve (fig. 2). This curve is approximately in a straight line ($E_1=2 \times Z/p$) from the point $E_1=2$ volts

Fig. 2.—Helium.



to $E_1=4$ volts. The energy of agitation is 4.7 volts at the point where $Z/p=2.6$, which is much below the 5.2 volts to be expected if λ were constant.

These results may be explained by the theory of ionization by collision as given by the author many years ago. In some of the collisions with the larger velocities of agitation the atoms acquire energy from the electrons in large amounts which may cause them to radiate, and in others the atoms may be ionized. But the transfer of energy in amounts which produce these effects occurs only in a small proportion of the total number of collisions with large energies of agitation.

3. This theory of ionization by collision gives a satisfactory explanation of luminous discharges which are maintained when the ratio of the electric force to the pressure of the gas is comparatively small*.

This may be seen in a direct-current discharge in a long, wide tube containing gas at a few millimetres pressure. In the uniform positive column of the discharge where the current is maintained by the ionization of the gas the electric force is comparatively small.

Recent experiments† with high-frequency discharges of constant amplitude have also shown that the currents in long tubes which are accompanied by a uniform luminous glow are also maintained with small electric forces.

The theory which has been given to account for the long luminous columns indicates that positive ions and electrons are lost by diffusion to the sides of the tube, and these losses are balanced by the supply provided by the process of ionization by collision. In wide tubes containing gases at a few millimetres pressure the losses due to diffusion are small, so that the supply required to maintain continuity is provided by the ionization of the gas obtained with a small electric force‡. The theory shows that the mean force, in a high-frequency discharge of constant amplitude, is the same as the force in the direct-current discharge. This result has been obtained experimentally with nitrogen§ and neon, but the experiments with neon have not yet been completed for publication.

4. The forces in the uniform luminous columns of high-frequency discharges have been determined by Hayman|| for neon and helium at various pressures in a tube 2.9 cm. in diameter.

The mean force Z which is independent of the current is given in volts per centimetre, and the pressure p in millimetres of mercury in Table I. for discharges in neon, and in Table II. for discharges in helium. The ratios Z/p are given in the third line of each table, and the

* 'Electricity in Gases,' p. 440. Clarendon Press, Oxford (1915).

† J. S. Townsend and R. H. Donaldson, *Phil. Mag.* v. p. 178 (Jan. 1928).

‡ *Comptes Rendus*, t. 186, p. 55 (9th Jan., 1928).

§ J. S. Townsend and W. Nethercot, *Phil. Mag.* vii. p. 600 (March 1929).

|| R. L. Hayman, *Phil. Mag.* vi. p. 586 (March 1929).

values of E_1 in volts corresponding to these values of Z/p , as obtained from the curves figures 1 and 2, are given in the fourth line.

These experiments were made in a quartz tube with external electrodes, so that it was possible to heat the whole tube to a very high temperature in order to remove impurities. The gases were carefully prepared, and the tube was washed out several times with pure gas before the measurements of forces in the discharges were made. As far as could be seen with a direct-vision spectroscope, there were no impurities in the gases. The light from

TABLE I.—Neon.

p	60	40	20	10	5	2
Z	28.5	21	11.5	7.0	4.4	3.5
Z/p47	.52	.57	.7	.88	1.75
E_1	4.8	5.2	5.6	6.4	7.5	—

TABLE II.—Helium.

p	20	10	5	2	1
Z	32	19	11.5	6.4	5
Z/p	1.6	1.9	2.3	3.2	5
E_1	3.2	3.8	4.3	5.3	6.4

the electrodeless discharge was examined in each experiment, as this method of detecting impurities was found to be much more sensitive than the ordinary method with a small spectroscopic tube*. These determinations of the forces in luminous discharges in pure gases are therefore very reliable.

In the discharge in neon at 20 mm. pressure the ratio of the force to the pressure is 0.57, and the mean energy of agitation of the electrons is 5.6 volts. This is about the point where losses of energy in large amounts have a noticeable effect on the mean energy of agitation in neon, as shown by the curve (fig. 1).

* J. S. Townsend and S. P. MacCallum, *Phil. Mag.* v. p. 695 (April 1928).

In the discharge in helium at 5 mm. pressure the ratio of the force to the pressure is 2.3, and the mean energy of agitation of the electrons is 4.3 volts, which is near the point where large losses of energy in helium are shown by the curve (fig. 2).

5. It does not appear that these experiments can be explained by any hypothesis in agreement with the laws governing the impact of an electron on a molecule of a gas that have been given by Franck and Hertz. These laws, as described by Jeans* and Atkinson†, are said to be very accurate and satisfactory, and are adopted by many writers, without questioning their validity.

There are, however, many phenomena which conflict with these laws of impact, and, if the methods by which they have been deduced from experimental data be examined, it is found that the evidence in support of them is not convincing.

The losses of energy which occur in the collisions of electrons with atoms of helium when the kinetic energy of the electrons is below a certain critical value has been estimated by finding the amount of energy lost in a comparatively small number of collisions. In these experiments the electrons move in an accelerating field, and lose energy in collisions with atoms of the gas while traversing a certain distance in the direction of the electric force. The experiments show that a small proportion of the total number of electrons lose a small amount of energy in traversing the gas, and this result is interpreted on the hypothesis that all the other electrons lose exactly the same amount as those comprised in the group which were found to have lost a small amount. The possibility of losses being distributed about a mean value is not considered, although, according to the ordinary dynamical theory of gases, there are many causes which give rise to inequalities in the amounts of energy lost by different groups of electrons in moving through the same distance in a gas. Also, if some electrons had lost energy in collisions with atoms in very large amounts, these losses would not have been detected. These experiments, however, are frequently quoted as proving that the loss of energy of electrons in collisions with atoms of monatomic

* Jeans, 'Report on Radiation and the Quantum Theory,' 2nd ed. (1924).

† Atkinson, Proc. Roy. Soc. A, cxix, p. 335 (1928).

gases is *exactly* the same as if the atoms were perfectly elastic, when the energy of an electron is below a certain value. This critical value is stated to be 19.77 volts in helium and approximately 16 volts in neon.

What has been measured experimentally is a lower limit to the amount of energy that electrons may lose under certain conditions; and if the result were reliable, there would be an upper limit to the energy that electrons can acquire while moving in helium under the action of an electric force. If the gas were at 5 mm. pressure and the force 11.5 volts per cm., this upper limit to the energy of agitation would be about 4.6 volts. It is easy to see that this interpretation conflicts with the results of many simple experiments.

6. The other laws governing the impacts of electrons with atoms relate to the large amounts of energy that electrons may lose in collisions. When the energy of an electron exceeds a certain amount, it may lose a large proportion of its energy in a collision with an atom.

According to the laws of impact now adopted by many writers, these losses of energy in large amounts occur immediately the kinetic energy of an electron attains a certain exact value. This value corresponds to a certain critical potential, and for each gas there are said to be several critical potentials which have been determined to a high degree of accuracy.

The critical potentials in helium and neon, and the properties of the gases with which they have been associated, are briefly as follows. In helium the lowest critical potential is 19.77 volts and the ionizing potential is about 25 volts. In neon the lowest critical potential is 16 volts and the ionizing potential 21 volts. In consequence of the absorption of energy in amounts corresponding to some of the critical potentials, the atoms are caused to radiate, and these potentials are called resonance potentials. The others are called metastable potentials. When the atoms absorb energy in the amounts corresponding to these potentials they are thrown into a metastable state. The metastable atoms do not radiate, but possess the property of ionizing molecules of impurities which may be present in the gas.

This process of ionization is known as the "Stossezweiter Art," and it is supposed to be very effective, especially in

helium, as the first critical potential in that gas is a metastable potential. In neon there are several critical potentials nearly the same as the first critical potential, and there seems to be some uncertainty as to whether the first critical potential is a metastable potential or a resonance potential.

As far as this discussion is concerned, it is of no importance whether the first critical potential is a resonance potential or a metastable potential.

The first resonance potential corresponds to a line in the ultra-violet spectrum, and according to the theory the full number of lines in the visible spectrum is not attained until the electrons acquire energy in amounts which may be 1 or 2 volts less than the ionizing potential. In both gases the ionizing potential is said to be about 5 volts greater than the first critical potential, so that when the energy of an electron increases gradually as it moves under the action of an electric force, it may lose all its energy in collisions with atoms in amounts corresponding to the first critical potentials before it attains the energy corresponding to one of the higher critical potentials. Under certain circumstances, therefore, it would be impossible to ionize the gas or to obtain lines in the visible spectrum. The ranges of the electric force and gas-pressure in which this result may be expected depend on the probability of an electron losing its energy in a collision, and on the difference between the ionizing potential and the first critical potential.

In the experiments on which the theory is based, changes in currents are obtained in helium with potentials of about 20 volts and in neon with potentials of 16 volts, and opinions have differed from time to time as to whether these effects were due to ionization by collision or to radiation from the gas, or to impurities. The point on which the advocates of this theory have always agreed is that the atoms of helium have a remarkable power of absorbing energy in amounts of about 20 volts, and the atoms of neon in absorbing energy in amounts of 16 volts, so that when the kinetic energy of the electrons just exceeds these critical amounts, very few can retain their energy after a few collisions with atoms of the gas.

There now seems to be some agreement on this point, and these potentials are said to be either metastable

potentials or resonance potentials, and the ionizing potentials are about 5 volts greater in each gas.

7. If these laws be true, there could be no appreciable effect due to ionization by collision or appreciable intensity in lines in the visible spectrum when the ratio Z/p is of the order of the values obtained in discharges accompanied by a uniform glow, in gases at a few millimetres pressure.

This may be seen by estimating the number of electrons which obtain energies above 20 volts in helium, when the gas is at 2 mm. pressure and the electric force 5 volts per cm. In this case the mean energy of agitation of the electrons is about 5 volts.

The electrons that diffuse in the direction of the force acquire energies greater than the mean energy E_1 , but the number that acquire energies of 19 or 20 volts is very small, since they lose energy so rapidly in the amounts $\lambda \cdot E$, when E is four times the mean energy. These numbers are given by the formulæ for the distribution of the energy of electrons in terms of the mean energy of agitation. Also, the probability P that the energy of an electron may increase from 20 to 24 volts is very small, even when no collision results in a loss of energy greater than $\lambda \cdot E$. This probability depends on the number of free paths that the electron traverses after attaining the energy of 20 volts. After traversing 160 free paths P is about 1.5×10^{-5} , after 200 free paths P is about 4.5×10^{-5} , and after 240 free paths P is about 10^{-4} . The probability increases with the number of free paths, and attains a maximum value about 8×10^{-4} when the electrons have traversed 800 free paths. Thus the number of electrons that attain energies of 23 or 24 volts is very small even when there are no losses of energy in large amounts. These calculations also show that if there were losses of energy in large amounts in 10 per cent. of the collisions when the energy of an electron exceeds 20 volts, the energy that an electron could attain would be limited, and no appreciable number would attain energies of 21 to 22 volts. Under these conditions most of the lines in the visible spectrum would be absent, whether the gas were pure or impure.

8. It has been suggested that in these cases the conductivity is due to impurities which are ionized by the

process known as the "Stosse zweiter Art," even when the amount of impurity is so small that it cannot be detected spectroscopically. The theory is nevertheless unsatisfactory for many reasons*. It indicates that it would be impossible to obtain the ordinary lines in the visible spectrum of a monatomic gas at a few millimetres pressure, and if the gas were pure it would be impossible to have ionization. Also, the amount of impurity required to maintain the conductivity would be proportioned to the current, so that considerable amounts of impurity would be required to maintain currents of the order of 10 milliamps. Under these conditions, it would be necessary to suppose that there is a continuous supply of impurity to the gas in a discharge-tube, as it has been found that the current tends to remove impurities from monatomic gases.

It is thus quite clear that the laws of impacts of electrons on atoms of monatomic gas as given by Franck and Hertz are not in agreement with the electrical properties of currents in wide tubes as found experimentally. Also, as far as radiation from the gas is concerned, these laws are unsatisfactory, since they indicate that the lines in the visible spectrum of monatomic gases would not be excited, under the conditions in which bright luminous columns of gas are obtained in discharge-tubes.

It may also be seen that these modern theories of critical potentials are not based on reliable experimental evidence. The experiments which are quoted in support of them depend on the interpretation of effects observed at the boundary of a space containing a gas, when large currents of electrons are projected into the space through apertures in the boundary. It is supposed that the kinetic energies of the electrons in the collisions with molecules of the gas are the same as the kinetic energy with which the electrons enter the gas. No allowance is made for the effect of the negative charge in the gas due to the accumulation of electrons. This charge exerts a repulsive force which reduces the kinetic energy of the electrons so that the energy of the electrons when they collide with molecules of the gas is less than the energy at the boundary. Also, the reduction of the

* J. S. Townsend and S. P. MacCallum, *Proc. Roy. Soc. A*, cxxiv. (1929).

energy is continuous as the electrons move from the boundary into the gas, so that no large proportion of the electrons collide with any particular velocity. Under these conditions it cannot be maintained that the potentials of the electrons as they enter the gas represent amounts of energy coinciding with critical properties of molecules.

9. In the experiments which have been made to determine the mean energy of agitation the electrons move in a uniform electric field, and the currents are small, so that there is no appreciable charge in the gas. The electric force is therefore undisturbed by the current, and the mean energy of agitation E_1 is found accurately in terms of the ratio Z/p . These results give the mean energy of agitation in other experiments where the ratio Z/p can be found experimentally, but many phenomena which are observed in gases depend on the distribution of energy about the mean.

It is clear that in many discharges ionization by collision is due to electrons which have energies greater than the mean, and the experiments with electrodeless discharges also show that the colour of the discharge depends on the distribution of the energies of electrons. In neon the colour is red at the higher gas-pressures and changes to yellow as the pressure is reduced. In helium the colour is yellow at high pressures and becomes bright green as the pressure is reduced. Since the forces in these discharges have been determined experimentally, the changes in the intensities of the lines of the spectrum of the gas may be found in terms of the changes in the energy of agitation. The experiments show that the lines of long wave-lengths are very intense compared with those of short wave-lengths when the kinetic energy of the electrons is small, and as the kinetic energy increases, the intensities of short waves increase in comparison with the long waves.

These observations are in accordance with a theory of the radiation from a discharge which was well known many years ago.

It is therefore of interest to determine the rate at which electrons tend to acquire the mean energy of agitation, and the disturbing effects which cause some of the elec-

trons to have energies which exceed the mean energy by definite factors.

10. The kinetic energy of an electron in a gas depends on the amount of energy transferred in collisions and the number of collisions with molecules when the electron traverses a given distance in the direction of the electric force. Electrons may gain or lose energy in collisions, but the amounts of energy lost are generally greater than the amounts gained, so that the average effect of a large number of collisions may be represented by a loss of energy which is proportional to the kinetic energy E of the electron. If λE be the average loss of energy in a collision, the coefficient λ is a small fraction, and the amounts of energy that may be lost or gained in collisions is much greater than λE *. Thus different groups of electrons may lose amounts of energy which are widely different from the average loss of energy, in traversing a given distance in the direction of the electric force even if all the electrons collided with the same number of molecules.

Under some conditions there are considerable differences in the numbers of collisions of electrons with molecules of the gas, since electrons may traverse several consecutive free paths which on an average may be either longer or shorter than the mean free path (l). When the total number of collisions with molecules is not very large, there are large inequalities in the amounts of energy lost by electrons due to these causes.

11. In general, when electrons move in a field of force the motion is automatically regulated, so that in moving through a given distance in the direction of the force the number of collisions of an electron with molecules of a gas is proportional to the kinetic energy of the electron. Thus electrons with large energies of agitation tend to lose much more energy in moving through a given distance in the direction of the force than do those with small energies.

This controlling action has an equalizing effect which brings the mean energy of agitation of a group of electrons to a definite fixed value when the electrons move in a uniform electric field.

* Proc. Roy. Soc. A, cxx. p. 511 (1928).

The inequalities arising from variations of the free paths about the mean free path l and the variations of the losses of energy in collisions about the mean value $\lambda \cdot E_1$ are therefore counteracted when the electrons move in the direction of the electric force so that no large deviations from the mean energy of agitation can arise from these causes.

12. The following investigation of the motion due to diffusion shows that there are wide differences in the velocities of different groups of electrons in the direction of an electric force, which are accompanied by large inequalities in the amounts of energy lost by the different groups.

The electrons that diffuse in the direction of the electric force traverse a given distance in that direction in a shorter time, and have fewer collisions with molecules, than the groups that move with the average velocity in the direction of the electric force. Thus some electrons gain energy by diffusion in the direction of the electric force, while others lose energy by diffusion in the opposite direction. In fact, the controlling action which regulates the general motion and causes the energies of different groups to approach a definite mean value, is in reality an average effect, which determines a steady state of motion where there is a distribution of energy about the mean energy of agitation due to diffusion. This distribution is independent of differences in the energies that may arise from other causes, and would be obtained if all losses of energy in collisions were equal to the mean loss of energy, and all free paths equal to the mean free path.

The effect of the controlling action which tends to equalize the energies of electrons will be considered first, in order to estimate the rates at which the mean energy of a group of electrons tends to approach a certain definite value E_1 which depends on the ratio Z/p .

13. In order to simplify these investigations, it will be assumed that there are no large losses of energy in the collisions of electrons with molecules of the gas, in the amounts required to ionize atoms of the gas or to excite radiation. It will also be assumed that the energies of the electrons are large compared with the energy of agitation of a molecule of the gas; that the loss of energy

of an electron in a collision is $\lambda \cdot E$, where E is the energy of the electron and λ a constant; and that the mean free path L of an electron in the gas at unit pressure is also constant and the same for all values of E .

In order to calculate the energies of different groups of electrons, it is convenient to represent the mean energy E of a group in terms of the mean energy E_1 of all the groups corresponding to the steady motion in a uniform electric field. Within the ranges of the electric forces and gas-pressures where λ and L are constant, E_1 is proportional to the ratio Z/p . The number of collisions of electrons with molecules of the gas in the space between two planes perpendicular to the electric force Z may be represented in terms of the average number of collisions C of electrons in the steady state of motion, when the planes are at the distance apart (a) where $Zea = E_1$.

Let W_1 be the mean velocity of the electrons in the direction of the electric force, T the time a/W_1 , and U_1 the mean velocity of agitation in the steady state of motion. The total distance traversed by an electron moving with the velocity U_1 in the time T is, $U_1 a/W_1$, and this distance is $C \times l$, where l is the mean free path. The number C is therefore given in terms of U_1 and W_1 by the formula $C = U_1 a/W_1 l$.

For the purpose of these investigations it is not necessary to know the values of λ and C , but it is important to know the product $\lambda \cdot C$.

In the steady state of motion the mean energy of agitation of the electrons is constant, so that the energy (Zea) gained by the electrons in moving through the distance (a) in the direction of the force is equal to the energy lost in collisions which is approximately $\lambda \cdot C \cdot E_1$.

Since $Zea = E_1$, the relation between the constants λ and C is given approximately by the formula $\lambda \times C = 1$. The degree of accuracy of this formula depends on the distribution of the velocities of agitation about the mean velocity U_1 , and for first approximations λ may be taken to be $1/C$.

Hence, if the potential difference Za between two planes is equal to the energy E_1 expressed in volts, the average number of collisions of electrons with molecules of the gas in the space between the planes is the same for all values of Z and p , and approximately equal to $1/\lambda$.

14. The mean velocity W of a group of electrons in the direction of the electric force, which is proportional to Z , is given in terms of the mean velocity of agitation U by an equation of the form

$$W = \frac{Z}{p} \times \frac{e}{m} \times \frac{L}{U} \times h, \dots \dots \dots (1)$$

where h is a numerical constant depending on the distribution of the velocities about the mean value U . The value of h for the Maxwellian distribution is 0.815, U being the root of the mean square of the velocity ($U = \sqrt{2E/m}$). The distribution of the velocities of electrons about the mean velocity U is not the same as in the Maxwellian distribution, and in order to determine h it is necessary to find an appropriate formula for the distribution of the velocities.

Equation (1) shows that W is inversely proportional to U , so that when the energy E of a group of electrons changes as the group moves under an electric force through a gas, the product $U \times W$ is approximately constant and equal to $U_1 \times W_1$. In cases where the distribution of the energies about the mean energy changes as the electrons move through the gas, the product $U \times W$ is not accurately the same as $U_1 \times W_1$, since the factor h in equation (1) changes with the distribution. This applies to a group which begins to move through a gas when the energies of the electrons are approximately the same, since the distribution of the energy in the first stages of the motion is not the same as in the final stages. For this reason the formulæ which are obtained on the hypothesis that the product $U \times W$ is a constant are only approximate, though in some cases the results approach a high degree of accuracy.

15. If the kinetic energy of a group of electrons be small, the energy increases as the electrons move through the gas*.

Let the electrons start from the plane $z=0$, and let E be the mean energy of agitation of the group in passing through a plane perpendicular to the electric force at a distance z from the origin. Also let dc be the average

* *Proc. Roy. Soc. A*, lxxxi. p. 461 (1908).

number of collisions of an electron with molecules of the gas in the space between the two planes z and $z+dz$. The electrons traverse the space dz in the time dz/W , and the number of collisions is proportional to Udz/W . Since C is the average number of collisions in the space between two planes at a distance apart a when the velocities are U_1 and W_1 , the number dc is given by the formulæ

$$\frac{dc}{C} = \frac{dz}{a} \cdot \frac{U}{U_1} \cdot \frac{W_1}{W} = \frac{E}{E_1} \cdot \frac{dz}{a}, \dots \quad (2)$$

and the loss of energy in the collisions is

$$\lambda \cdot E \, dc = E^2 dz / a E_1. \dots \quad (3)$$

The rate of increase of E with z is given by the equation

$$\begin{aligned} dE &= Ze \, dz - E^2 dz / a E_1 \\ &= Ze(1 - E^2/E_1^2) \, dz, \dots \quad (4) \end{aligned}$$

which on integration gives

$$\log (E_1 + E) / (E_1 - E) = 2Ze z / E_1 = 2z/a. \dots \quad (5)$$

This equation is represented by Curve 1 (fig. 3), the ordinates being the ratios E/E_1 and the abscissæ the ratios z/a .

If the energy of agitation E be greater than the energy corresponding to steady motion, the electrons lose energy as they move through the gas. The relation between E and z is then given by the equation

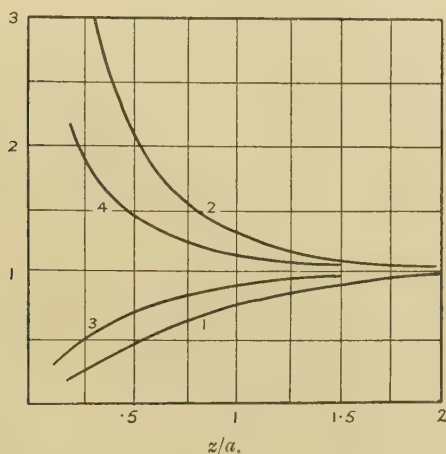
$$\log (E + E_1) / (E - E_1) = 2z/a, \dots \quad (6)$$

the constant of integration being chosen to make E very large compared with E_1 at the plane $z=0$.

Equation (6) is represented by Curve 2 (fig. 3). It will be observed that the ordinates of Curve 2 are the reciprocals of the ordinates of Curve 1. The curves show that if the mean energy of the electrons in one group be $0.25E_1$ and in another group $4E_1$ at the plane $z=0.255a$, the energy of the first group increases and that of the second group diminishes as the electrons move in the direction of the electric force, so that at the plane $z=0.55a$ the energies are $.5E_1$ and $2E_1$, and at the plane $z=2a$ the energies are $.96E_1$ and $1.04E_1$ in the two groups respectively.

16. The velocities of agitation of the two groups in passing a plane at the distance z from the origin are given by Curves 3 and 4 (fig. 3), the ordinates being the ratios U/U_1 and the abscissæ the ratios z/a . These ordinates are the square roots of the ordinates of Curves 1 and 2 respectively. The time required for a group to traverse the distance dz is inversely proportional to W and therefore directly proportional to U . Hence the area between Curve 3 and the ordinates at the distances z_1 and z_2 from the origin is proportional to the time in which the first group moves through the distance $(z_2 - z_1)$, and the

Fig. 3.



Ordinates of Curves 1 and 2, E/E_1 .
Ordinates of Curves 3 and 4, U/U_1 .

corresponding change in energy is given by the ordinates of Curve 1 at the points z_1 and z_2 .

Similarly the area between Curve 4 and the ordinates of two points of the curve gives the time in which a change of energy takes place in the second group. The unit of area represents the time $T = a/W_1$. The curves show that the energy of the first group increases from $0.25E_1$ to $0.5E_1$ in the time $0.18 \times T$, and the energy of the second group diminishes from $4E_1$ to $2E_1$ in the time $0.5 \times T$.

The velocities W for the two groups are given in terms of W_1 by Curves 3 and 4, the ordinates being the ratios W/W_1 for the groups with energies given by Curves 2 and 1 respectively.

Hence, in general when the energies of groups of electrons become large or small compared with the energy E_1 , changes arise in the motion which tend to equalize the energies, so that the groups have approximately the same mean energy E_1 after traversing the distance $2a$ in the direction of the electric force.

17. In order to determine the effect of diffusion in causing inequalities in the energies of the electrons, the inequalities arising from variations of the free paths about the mean free path (l) and variations of the losses of energy in collisions about the mean loss λE_1 may be left out of consideration.

In the actual motion in a field of force, different groups of electrons move with different velocities in the direction of the electric force. The distance that a group moves in that direction in the time dt is $(W+w')dt$, W being the mean velocity due to the electric force and w' the velocity due to diffusion. The velocity w' is different for different groups of electrons, and in some groups containing small numbers of electrons w' may be very large compared with W for short intervals of time.

The motion of electrons in the direction z due to diffusion may be found by the theory of diffusion of particles where the coefficient of diffusion is taken to be constant. The space distribution at a given time may be expressed in terms of the number of free paths c traversed by the particles, and for this investigation it is convenient to express c in terms of the number C and the distances z in terms of a , where a is $E_1/Z.e$.

The ordinary equations of motion of electrons in an electric field of intensity Z may be written in the form

$$\frac{pw}{K} = -\frac{dp}{dz} + neZ, \quad . \quad . \quad . \quad . \quad (7)$$

where K is the coefficient of diffusion, n the number of electrons per cubic centimetre, and p the partial pressure of the electrons.

In the steady state of motion where E_1 is the mean energy of agitation the partial pressure p is $2nE_1/3$, and w is the velocity W_1 due to the electric force when dp/dz is zero.

Equation (7) therefore gives

$$K_1/W_1 = 2E_1/3Ze = 2a/3, \quad . \quad . \quad . \quad . \quad (8)$$

and the product $4K_1T$ is therefore $8a^2/3$, T being the time a/W_1 , and K_1 the mean coefficient of diffusion in the steady motion. This product may also be expressed in terms of the mean free path l , $4KT=4l^2C/3$, C being the number of free paths traversed by each electron in the time T .

Hence, if c be the number of free paths traversed in the time t , the product $4K_1t$ is given in terms of the ratio c/C by the formula

$$4K_1t = 8a^2c/3C \quad . \quad . \quad . \quad . \quad . \quad (9)$$

or

$$4K_1t = A c/C,$$

where A is $8a^2/3$ or $4l^2C/3$.

18. The effect of diffusion in causing inequalities in the energies of agitation may be deduced from the space distribution due to diffusion when there is no electric force acting on the electrons. All the electrons may be supposed to be at the plane $z=0$ at the time $t=0$, and it is required to find the motion due to diffusion when all the electrons are moving with a constant velocity of agitation. Under these conditions the coefficient of diffusion is constant and the same for each electron, and K may be taken to be equal to the mean coefficient of diffusion K_1 in the steady state of motion.

The space distribution is thus obtained from the equation

$$\frac{dn}{dt} = K \cdot \frac{d^2n}{dz^2}, \quad . \quad . \quad . \quad . \quad . \quad (10)$$

ndz being the number of electrons between the planes z and $z+dz$ at the time t .

The solution of this equation, which satisfies the initial conditions, is

$$n/N = (4\pi Kt)^{-\frac{1}{2}} e^{-z^2/4Kt}, \quad . \quad . \quad . \quad . \quad (11)$$

which gives $n=0$ for all values of z when t is zero except at the plane $z=0$. In this equation N is the total number of electrons which are near the plane $z=0$ at the time $t=0$.

The ratio n/N is obtained in terms of z , and the number of free paths c traversed by the electrons in the time t by substituting in equation (11) the value of $4Kt$ given by equation (9). Hence

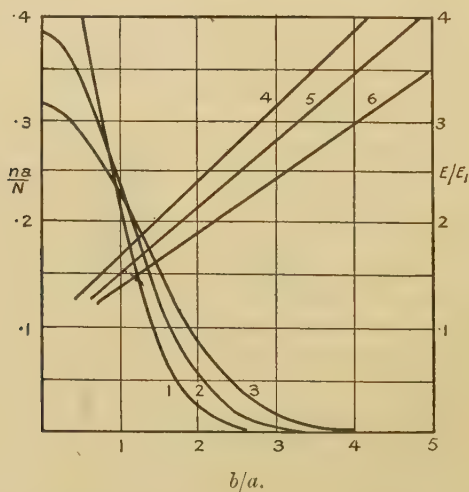
$$n/N = (8\pi l^2 c^2/3C)^{-\frac{1}{2}} e^{-3z^2C/8a^2c} \quad . \quad . \quad . \quad (12)$$

19. The notation is simplified by writing $f(t, z)$ and $f(c, z)$ for the expressions on the right of equations (11) and (12) respectively.

Equation (11) gives the distribution at a given time t when the velocity of agitation is constant; but it is inconvenient to express the distribution in terms of t , since the energy of agitation of different groups of electrons changes when the motion takes place in an electric field.

Equation (12) gives the distribution after each electron has traversed a given number of free paths c . In this equation the distance z is the sum of the projections on

Fig. 4.



Ordinates of Curves 1, 2, 3, na/N .

Ordinates of Curves 4, 5, 6, E/E_1 .

Curves 1 and 4, $C_1 = .5 C$.

Curves 2 and 5, $C_1 = .8 C$.

Curves 3 and 6, $C_1 = 1.2 C$.

the axis of z of the free paths of different electrons, the free paths being straight lines. This distribution is independent of the velocity of agitation, and equation (12) may be interpreted as giving the probability of the sum of the projections of the free paths being between the lengths z and $z+dz$. Equation (12) is therefore very convenient for determining the numbers in different groups when the motion takes place in an electric field.

The distribution given by equation (12) may be represented by a curve giving the positions of the electrons after traversing the number of free paths c . It is convenient to change the letter z in this equation to b , in order to indicate the distances of the electrons from the plane $z=0$, when each electron has traversed a given number of free paths c_1 . The following expression is thus obtained for the ratio na/N ,

$$na/N = .346 \sqrt{C/c_1} \times e^{-b^2 C / 8a^2 c_1} \dots (13)$$

This equation is represented by Curves 1, 2, and 3 (fig. 4), which correspond to the three values of c_1 , .5C, .8C, and 1.2C. (In neon the number C is approximately 1.4×10^4 , and in helium 4×10^3 .)

The ordinates of the curves are the ratios na/N^3 , and the abscissæ the ratios b/a .

The curves show that the number of electrons that attain distances b greater than $4a$ from the plane $z=0$ is very small. When c_1 is 1.2C the number that exceed this distance is 8×10^{-4} .

The mean distance \bar{b} from the plane $z=0$ of the electrons ($N/2$) on one side of the plane is

$$\bar{b} = 2 \int_0^\infty b f(c_1, b) db = a \sqrt{8c_1} / 3\pi C \dots (14)$$

When c_1 is equal to C, \bar{b} is $.92 \times a$.

20. The electrons which diffuse from the plane $z=0$ may be divided into groups which are at different distances b from the plane $z=0$ when the number of free paths c_1 have been traversed. In order to estimate the loss of energy in collisions, it is necessary to find the mean distance of a group from the plane $z=0$ when a proportion c/c_1 of the free paths have been traversed by each electron. Let $n_1 db$ be the number of electrons in the group between the planes $z=b$ and $z=b+db$ after the number of free paths c_1 have been traversed, $n_1 db = N \times f(c_1 b) db$; also let $c+c'=c_1$.

The number of electrons in the space between the planes z and $z+dz$ after the total number N have traversed the number of free paths c is $ndz = N \times f(c, z) dz$, which includes some of the electrons belonging to the group $n_1 db$. The number that belong to the group $n_1 db$ is obtained by multiplying ndz by $f(c', (b-z)) db$, since the

electrons that belong to the group $n_1 db$ must be at the distance $(b-z)$ from the plane z when the remainder c' of the free paths have been traversed.

Let $z=(bc/c_1+x)$, $(b-z)=(bc'/c_1-x)$, and $dz=dx$. When these values are substituted for z , $b-z$, and dz in the above expressions, the number of electrons belonging to the group $n_1 db$ which are between the planes z and $z+dz$ after traversing the number c of free paths is

$$N dx db \times f(c, (cb/c_1+x)) \times f(c', (c'b/c_1-x)).$$

This expression may be written in the form

$$N \times (\pi A c_1)^{-\frac{1}{2}} e^{-b^2/4Ac_1} db \times (\pi A c c'/c_1)^{-\frac{1}{2}} e^{-c_1 z^2/4cc'} dx,$$

which reduces to

$$n_1 db \times f(cc'/c_1, x) dx. \quad . \quad . \quad . \quad (14)$$

These expressions show that the group $n_1 db$ is distributed symmetrically about the plane $z=cb/c_1$ when the electrons have traversed the number of free paths c . Thus $dz=bdc/c_1$, z being the mean distance of the group from the plane $z=0$.

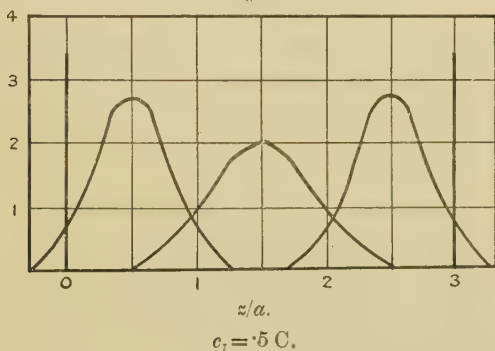
Also the distribution of the group about the central plane is the same as that attained when electrons diffusing from the plane have traversed the number of free paths cc'/c_1 . Thus as the group moves away from the plane $z=0$ the mean distance from the central plane increases and attains the maximum value $a \sqrt{2c_1/3\pi C}$, when $c=c_1/2$, and the central plane is at the distance $b/2$ from the plane $z=0$. When the group passes through this position, the mean distance of the group from the central plane diminishes, and the electrons collect together as the central plane approaches the plane $z=b$.

21. The above results may also be obtained by taking the distributions as given in terms of the time t by equation 11, when the velocity of agitation is constant. The group of electrons $n_1 db$ which are between the planes $z=b$ and $z=b+db$ at the time t_1 is symmetrically distributed about the plane $z=tb/t_1$ at any time t less than t_1 . Thus the mean velocity of the group is a constant and equal to b/t_1 while passing from the plane $z=0$ to $z=b$. At any time t greater than t_1 all directions of motion of the electron in the group $n_1 db$ are equally probable, and the mean velocity of the group due to diffusion is zero.

These results are represented by the curves (fig. 5) which give the distributions of the group $n_1 db$, in three positions between the planes $z=0$ and $z=b$. The distance b is taken as being $3a$ and the total number of free paths c_1 as $C/2$.

The ordinates of the curves are proportional to the numbers of electrons at the planes z , and the abscissæ are the ratios z/a . Curve 1 represents the distribution when the number of free paths c traversed by the electrons is $c_1/6$, and Curve 3, which is similar to Curve 1, gives the distribution when c is $5c_1/6$. The mean distance of the distribution from the central plane in these positions is $x=0.24 \times a$. Curve 2 represents the distribution when c is $c_1/2$, the central plane being at the distance $1.5a$ from

Fig. 5.



Motion of Group $n_1 db$.

the plane $z=0$. In this position the mean distance of the group from the central plane is $x=0.325a$.

22. When electrons start from a plane $z=0$ and move in an electric field, the change of energy of different groups of electrons depends on the rate of diffusion in the direction of the electric force. The change in energy may be estimated in terms of the distance b that the electrons diffuse in the direction of the electric force while traversing a given number of free paths c_1 . If all the electrons (N) start with the same kinetic energy E_1 , the numbers $n_1 db$ in the groups that have approximately the same energy after traversing the given number of free paths c_1 are given by equation (12) ($n_1 db = N db f(c_1, b)$).

In order to find the rate of change of energy of the group ($n_1 db$), as the electrons move away from the plane $z=0$, let the central plane of the group move through the distance dz while the electrons traverse the number of free paths dc . The mean increase of energy dE of the group is therefore

$$dE = Ze dz - \lambda E dc, \quad . \quad . \quad . \quad (15)$$

at any time during the motion while c is less than c_1 .

The distance dz includes the distance bdc/c_1 that the electrons diffuse, and the distance that the electrons move owing to the action of the electric force. The latter distance is the distance dz given by equation (2) in terms of dc , and is equal to $aE_1 dc/EC$.

Thus the distance dz in equation (15) is

$$dz = b dc/c_1 + aE_1 dc/EC. \quad . \quad . \quad . \quad (16)$$

Hence dE may be expressed either in terms of dz or dc . The latter is the most convenient variable to take, since the limits of integration are from $c=0$ to $c=c_1$, and the number n_1 is expressed in terms of c_1 by equation (12).

Equation (15) thus becomes

$$dE = Ze(b/c_1 + aE_1/EC) dc - E dc/C$$

or

$$EdE = (EE_1 bC/ac_1 + E_1^2 - E^2) dc/C. \quad . \quad . \quad (17)$$

Let the constant bC/ac_1 be represented by the expression $(y-1/y)$, where y is greater than unity, so that equation (17) becomes

$$EdE = (E_1 + yE)(E_1 - E/y) dc/C, \quad . \quad . \quad (18)$$

which on integration gives

$$\frac{y}{y^2 + 1} \left(y \log \left(\frac{yE_1}{yE_1 - E} \right) - \frac{1}{y} \log \left(\frac{E_1 + yE}{E_1 + yE_1} \right) \right) = \frac{c}{C}, \quad (19)$$

E_1 being taken as the value of E when $c=0$.

This equation gives the values of E for the electrons in the group $n_1 db = Ndbf(c_1, b)$ in terms of c , the number c_1 which occurs in the constant bC/ac_1 being the maximum value of c .

23. Equation (19) may be represented by a curve giving E/E_1 in terms of c/C for a group corresponding to a given value of y . The curves thus obtained for eight

different values of y (or eight different groups of electrons) are given in fig. 6, the ordinates being the ratios E/E_1 and the abscissæ the ratios c/C .

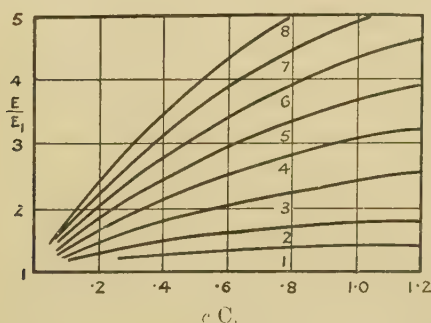
The values of y are 1.5, 2, 3, etc. . . . 8, and the corresponding values of bC/ac_1 for the eight groups are (1.5—.66), (2—1/2), (3—1/3), and . . . (8—1/8).

The values of E/E_1 for the different groups when all the electrons have traversed a given number of free paths c_1 are given by the ordinates of the eight curves at the points when the abscissæ are equal to c_1/C , and the values of b/a for the different groups are obtained from the equation

$$b/a = (y - 1/y)c_1/C. \quad . \quad . \quad . \quad (20)$$

The results thus obtained when c_1 is .8C are given in Table III.

Fig. 6



A similar set of figures may be obtained for any other value of c_1 from the curves (fig. 6).

24. The values of E/E_1 corresponding to the three values of c_1 , .5C, .8C, and 1.2C are given in terms of the ratio b/a by Curves 4, 5, and 6 (fig. 4).

The ordinates are the ratios E/E_1 , and the scale is given by the figures to the right of the diagram. The abscissæ are the ratios b/a , the scale being the same as for Curves 1, 2, and 3.

Thus the pair of Curves 1 and 4 give the numbers of electrons and the energies of the electrons in the different groups when the number of free paths traversed by each

electron is 0.5 C. Similar results are given by the pair of Curves 2 and 5 when the number of free paths traversed by the electrons is 0.8 C, and by the pair of Curves 3 and 6 when the number of free paths is 1.2 C.

The curves, 4, 5, and 6 are almost exactly straight lines for the values of the ratio E/E_1 exceeding 1.2.

The equations of the lines are

$$\left. \begin{array}{l} \text{Curve 4: } c_1 = .5 \text{ C, } E/E_1 = .95 + .74 b/a, \\ \text{Curve 5: } c_1 = .8 \text{ C, } E/E_1 = .85 + .65 b/a, \\ \text{Curve 6: } c_1 = 1.2 \text{ C, } E/E_1 = .85 + .53 b/a. \end{array} \right\} \dots (21)$$

The agreement between these expressions for E/E_1 , and the values of E/E_1 obtained from Curves (fig 6),

TABLE III.—($c_1 = .8 \text{ C}$).

y .	b/a .	E/E_1 .	$.85 + .65b/a$.
1.5.....	.66	1.32	1.28
2.....	1.20	1.65	1.63
3.....	2.14	2.25	2.24
4.....	3.0	2.80	2.80
5.....	3.84	3.34	3.34
6.....	4.67	3.86	3.88
7.....	5.5	4.4	4.42
8.....	6.3	4.95	4.93

is shown by the numbers in the last column of Table III. These numbers are the values of $(.85 + .65b/a)$, and are almost exactly the same as the values of E/E_1 given in the third column of the table.

25. The equations of Curves 1, 2, and 3 (fig. 4) are given by equation (13). Thus the equation of Curve 2, where $c_1 = 0.8 \text{ C}$, is

$$na/N = .385 \times e^{-L^2/a^2 \times 2.13}, \dots (22)$$

and the number of electrons $n_1 db$ in the group where the abscissæ are between b/a and $(b+db)/a$ is

$$n_1 db = 0.385 N \times e^{-b^2/a^2 \times 2.13} db/a. \dots (23)$$

Let Δn be the number in the group where the energies are between the values E and $E+dE$ after each electron has traversed the number of free paths c_1 . Since the

values of b/a are given in terms of E/E_1 by equations (21), the number Δn may be expressed in terms of E and dE .

When c_1 is $\cdot 8C$, the expression for ΔN obtained by substituting $(E/E_1 - \cdot 85)/\cdot 65$ for b/a on the right of equation (23) is

$$\Delta N = 0\cdot 6 \times N \times e^{-1\cdot 1(E/E_1 - \cdot 85)^2} \times dE/E_1. \quad (24)$$

The formulæ corresponding to values of c_1 from $\cdot 8C$ to $1\cdot 2C$ are exactly the same. Thus there is a considerable range of values of c_1 where the distribution is constant, so that in the steady state of motion the numbers of electrons with energies exceeding $1\cdot 2E_1$ are given by equation (24).

26. In the initial stages of the motion, when a number of electrons begin to move under the action of an electric force with the same kinetic energy E_1 , very few electrons attain energies which are much greater than E_1 . This investigation shows that the distribution given by equation (24) is not fully attained at the stage where all the electrons have traversed the number of free paths $\cdot 5C$. In the formula for ΔN corresponding to the value of $c_1 = \cdot 5C$ the logarithm of the exponential term is $-1\cdot 36(E/E_1 - \cdot 95)^2$, so that for the larger values of E/E_1 the number ΔN is less than that obtained when c_1 is $0\cdot 8C$.

For values of c_1 that are large compared with C , the groups with large energies are not separated in space from those with smaller energies, so that the distribution of energy is not represented in terms of b/a by equation (12). This is seen from the formulæ for ΔN , corresponding to values of c_1 greater than $1\cdot 2C$. The divergence from equation (24) begins to be appreciable when c_1 is $1\cdot 4 \times C$, the logarithm of the exponential term being $1\cdot 1(E/E_1 - 0\cdot 75)^2$ for this value of c_1 .

27. The investigation shows that the final distribution is a function of the ratio E/E_1 , which is the same for all gases and independent of the force Z and the pressure p of the gas, since the values of λ , l , Z , and p are not involved in the formulæ for ΔN .

28. It is convenient to have a simple method of finding the proportion of the electrons with a given energy $x_1 E_1$ that acquire energies greater than $x_2 E_1$ after each electron

has traversed a given number of free paths c_1 , when $(x_2 - x_1)$ is small compared with $(x_1 + x_2)$. Let N be the total number of electrons with the energy $x_1 E_1$, N_1 the number that acquire energies greater than $x_2 E_1$, and let the motion along the free paths begin at the plane $z=0$. The distribution due to diffusion after each electron has traversed the given number c_1 of free paths is given by equation (13) in terms of the ratio b/a . The electrons which acquire energies greater than $x_2 E$ are at distances b from the plane $z=0$, which exceed a certain value.

The number N_1 is found by integrating ndb (equation (13)); and since this quantity is given in the form $N\pi^{\frac{1}{2}}e^{-s^2}ds$, N_1/N is obtained from tables of the integrals of the function $e^{-s^2}ds$.

The value of b/a for the group which attain the energy $x_2 E_1$ is obtained from equation (17) by taking dE to be $(x_2 - x_1)E_1$, dc to be c_1 , and E to be the average energy $(x_1 + x_2) E_1/2$. The following equation is thus obtained:

$$b/a = x_2 - x_1 + ((x_1 + x_2)/2 - 2/(x_1 + x_2))c_1/C, \quad (25)$$

which gives the lower limit of b/a in the integral

$$\int_b^\infty ndb = N_1.$$

The ratio N_1/N increases with c_1 , when c_1/C is a small fraction, and attains a maximum value for a certain value of c_1 depending on x_1 and x_2 . Equation (13) shows that N_1 depends on the quantity $b^2 C/a^2 c_1$ which occurs in the exponential term, and the minimum value of this quantity as obtained from equation (25) is

$$(b^2 C/a^2 c_1) = 2(x_2 - x_1)(x_1 + x_2 - 4/(x_1 + x_2)), \quad (26)$$

the corresponding value of c_1/C being

$$(c_1/C) = (x_2 - x_1)/(x_1 + x_2)/2 - 2/(x_1 + x_2)$$

and

$$(b/a = 2(x_2 - x_1).$$

As an example of these formulæ, the electrons may be supposed to be moving in helium where the number C is approximately 4000, and that the energy of a group of N is $4E_1$. After each electron in this group has traversed 200 free paths, $c_1/C = 1/20$, the number N_1 that have acquired energies greater than $5E_1$ is $1.6 \times N \times 10^{-6}$.

The number N_1 increases with c_1 and attains a maximum value $2 \times N \times 10^{-4}$ when c_1/C is $1/4.3$.

29. The accuracy of the formulæ for the distribution of the energy E about the mean energy E_1 obtained by this method depends principally on the accuracy to which the product $\lambda \times C$ may be found. A rough calculation shows that λ is less than $1/C$, and that the relation between λ and C is given more accurately by the formulæ $\lambda = 1/1.1 \times C$. As a first step towards finding a more accurate expression for the distribution of the energy, λ may be taken to be $1/1.1 \times C$ instead of $1/C$, and this has the advantage of giving an expression for ΔN which is simpler than the expression in the formulæ (24). This change affects equations (16) to (21), since they were obtained by substituting $1/C$ for λ in equation (15). If $1.1 C$ be substituted for C in these equations, it is found that the value of $b/a = (E/E_1 - .85)/.65$ corresponds to the number of collisions c_1 which is given in terms C by the formulæ $c_1 = 0.88 C$. When these values of b/a and c_1 are substituted in equation (13), the following equation for ΔN is obtained instead of equation (24):

$$\Delta N = N \pi^{-\frac{1}{2}} e^{-(E/E_1 - .85)^2} dE/E_1. \quad . \quad . \quad . \quad (27)$$

The number of electrons (N, E) with energies greater than E is obtained by integrating equation (27). Table IV. gives the proportion $(N, E)/N$ of the total number with energies exceeding the mean energy E_1 by the factors given in the first line of the table.

TABLE IV.

$E/E_1 \dots$	1.5	2	2.5	3	3.5	4	4.5
$(N, E)/N \dots$	0.18	0.05	0.01	1.2×10^{-3}	9×10^{-6}	5×10^{-6}	1.5×10^{-7}

The figures show that only 1 per cent. of the total number N have energies greater than $2.5 E_1$. The average energy of the electrons with energies greater than E_1 is $1.5 E_1$ approximately.

There are other points of interest in connexion with this method of investigating the distribution of energy which will be considered in another paper.

CV. *Elastic Impact of a Pianoforte Hammer.*

By R. N. GHOSH, D.Sc.*

[Plate XII.]

IN a series of papers † the author has shown that, in the case of felt hammers, the duration of contact of the hammer with the string, the displacement of the striking point, and the velocity of rebound can be calculated from the theory presented by him. In this theory the hammer was considered to be slightly elastic, and the free vibrations of the short portions of the string between the nearer fixed end and the striking point were considered to be negligible. Recent experiments ‡ show that slight modification is required in order that the theory may be applicable (1) when the free vibrations are finite, and (2) when the impact is very elastic.

It is found in case (1) that when the hammer is slightly elastic the pressure does not rise and fall continuously, but by jumps at τ , 2τ , 3τ , etc., where

$$\tau = \frac{2a}{c}.$$

where a = striking distance and c = transverse wave velocity.

The introduction of greater softness simply diminishes the sharp rise. For the sake of comparison two curves showing the relation between pressure and time during contact have been appended in this paper.

Previous Results.

The earliest theory was given by Helmholtz §, who assumed that the duration of contact was a very small fraction of the period of vibration of the whole string. Kaufmann ||, however, showed that the string is appreciably displaced during impact, and gave a theory for a hard hammer when the striking point was close to a fixed end. He assumed that the short portion of the string takes up a form during impact

$$y_1 = f(ct) \frac{x}{a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

* Communicated by Prof. M. N. Saha, F.R.S.

† Phys. Rev. xxviii. p. 458 (1926).

‡ Phil. Mag. vii. p. 346 (1929).

§ Ellis, 'Sensation of Tone' (Translation), p. 380.

|| Ann. der Physik, liv. p. 675 (1895).

and on the longer side

$$y_2 = f(ct - x + a). \quad \dots \quad (2)$$

Profs. Raman * and Banerji, however, gave a general theory for any striking distance, assuming that the form of the string is given by

$$\left. \begin{aligned} y_1 &= A_k \sin kx \sin k(l-a), & 0 < x < a \\ y_2 &= A_k \sin k(l-x) \sin ka, & a < x < l \end{aligned} \right\} \quad \dots \quad (3)$$

Equations (3) assume that the wave reflected from the further end reaches the striking point during impact, which experiment shows is not always true. The present author extended Kaufmann's theory to the case of an elastic hammer, and arrived at the following formula † :

$$\left. \begin{aligned} q^2 &= \frac{T_0}{aM} \frac{1}{X} - \frac{1}{4} k^2, \\ k &= \frac{T_0}{Mc} \frac{1}{X}, \\ X &= \left(1 + \frac{T_0}{\mu a}\right). \end{aligned} \right\} \quad \dots \quad (4)$$

Among other theories, Das's functional formulæ may be mentioned here. Dr. W. H. George has made a comparative study of other theories, and a complete account will be found in his paper ‡.

In the present paper modifications are introduced on account of free vibrations of the short portion of the string mentioned in the introduction.

Let us first take the case of a slightly elastic hammer of effective mass M and elasticity μ . Divide the total duration of impact into two parts, viz.: (1) the interval during which the hammer is in contact with the string but the latter is not displaced from its initial position, and (2) the interval when the string is shifted from its equilibrium position with a finite velocity. Let one end of the string be the origin of x axis and the striking distance be a . Let ζ be the displacement of the hammer measured from the point of contact; then during the first interval the equation of motion of the hammer is given by

$$M\ddot{\zeta} = -\mu\zeta. \quad \dots \quad (5)$$

* Proc. Roy. Soc. xcvii. p. 99 (1919).

† Phil. Mag. xlvii. p. 1141 (1924).

‡ Proc. Ind. Assoc. Calcutta, vii. p. 13, and subsequent papers (1920). Also Proc. Phys. Soc. London, xl. p. 30 (1927).

Hence

$$\xi = \frac{V_0}{n} \sin nt, \quad . \quad . \quad . \quad . \quad (6)$$

$n^2 = \frac{\mu}{M}$, and V_0 is the initial velocity of the hammer.

The pressure due to impact is given by

$$p_0 = V_0 (M\mu)^{\frac{1}{2}} \sin nt. \quad . \quad . \quad . \quad . \quad (7)$$

This state of affairs continues till a velocity wave of magnitude $V_1 \frac{\text{cm.}}{\text{sec.}}$ is produced. This happens when the pressure is

$$p_1 = \frac{2T_0 V_1}{c}, \quad . \quad . \quad . \quad . \quad (8)$$

c being the transverse wave velocity and T_0 the tension. And we find from (7) and (8) that the wave is generated at a time t_1 given by

$$\tan nt_1 = 2 \left(\frac{T_0}{\mu} \frac{\rho}{M} \right)^{\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad (9)$$

For

$$\frac{T_0}{\mu} = .2, \quad \rho = .028, \quad M = 6.6 \text{ gm.},$$

$$t_1 = 1.8 \times 10^{-5} \text{ sec.}$$

Total duration of impact $= .95 \times 10^{-2} \text{ sec.}$

During the second régime, we assume that the displacement of any point on the short portion of the string is given by

$$y_x = y_a \sum \frac{\sin kx}{\sin ka} + \sum P_r \sin \frac{r\pi x}{a} \sin \frac{r\pi ct'}{a}, \quad . \quad . \quad (10)$$

$t' = t - t_1$, y_a = displacement at $x = a$. While on the longer side

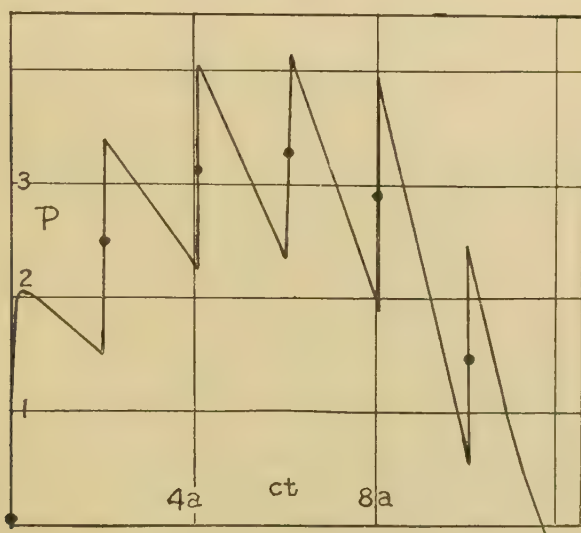
$$y_2 = f(ct' - x + a), \quad \text{and} \quad f(ct') = y_a. \quad . \quad . \quad (11)$$

The second term in (10) is due to the free vibrations of the short portion of the string between the nearer end and the striking point, which is considered to be a nodal point for all the free vibrations of that portion. This is justified on experimental grounds, which indicate that the small part of the string between the nearer fixed end and the striking point vibrates with the latter point as nodes for various modes. Fig. 1 shows the phenomenon. It will be seen that, in addition to the hump due to the impact of the hammer, there are smaller kinks superposed on the hump. The figure also shows that the free vibrations form a two-step

zigzag curve, and it has also been found that the period is nearly equal to $\frac{2a}{c}$. (See Table at the end of this paper.)

The first term is due to the hump, the values of k being unknown at present. Equation (11) assumes that the other portion of the string is too long for the reflected wave from the further end to arrive during the time of contact of the hammer with the string. This assumption is generally true in the actual case of the pianoforte where a/l varies from $\frac{1}{7}$ to $\frac{1}{32}$.

Fig. 1.



$$\text{Now} \quad \zeta = y + \xi \quad \text{and} \quad p = \mu \xi, \quad \quad (12)$$

$$M\ddot{\zeta} = -p.$$

Also

$$p = T_0 \left\{ \frac{ky_a}{\tan ka} + \frac{\dot{y}_a}{c} \right\} + T_0 \sum \frac{r\pi}{a} (-1)^r \sin \frac{r\pi ct'}{a}, \quad (13)$$

ζ = compression of felt.

From (12) and (13) we obtain the equation of motion of the striking point,

$$\begin{aligned} \frac{MT_0}{\mu c} \ddot{y}_a + M \left(1 + \frac{T_0 k}{\mu \tan ka} \right) \ddot{y}_a + \frac{T_0}{c} \dot{y}_a + \frac{T_0 k y_a}{\tan ka} \\ = \sum P_r T_0 \frac{r\pi}{a} (-1)^r \sin \frac{r\pi ct'}{a} \left\{ \frac{r^2 \pi^2 T_0 M}{a^2 \mu \rho} - 1 \right\}. \quad (14) \end{aligned}$$

We shall neglect the term in \ddot{y}_a as it is very small when μ is large, and solve for y_a by trial. We find that for small values of a , $\tan ka$ is small, and is approximately equal to ka . In a case when $a=15$ cm., and the other constants have the values as before, $k=0.0198$, $\tan ka=0.300$, and $ka=0.297$.

Thus for all practical purposes we may take $\tan ka=ka$, and then (14) reduces to

$$\begin{aligned} \frac{MT_0}{\mu c} \ddot{y}_a + M \left(1 + \frac{T_0}{\mu a} \right) \ddot{y}_a + \frac{T_0}{c} \dot{y}_a + \frac{T_0}{a} y_a \\ = \Sigma P_r \frac{r\pi}{a} (-1)^r T_0 \sin \frac{r\pi ct'}{a} \left\{ \frac{r^2 \pi^2 T_0 M}{a^2 \rho \mu} - 1 \right\}, \quad (15) \end{aligned}$$

and (10) takes the form

$$y_x = \frac{y_a x}{a} + \Sigma P_r \sin \frac{r\pi x}{a} \cdot \sin \frac{r\pi ct'}{a}.$$

Let us now determine the values of P_r before solving (15) completely. We observe that $t'=0$,

$$\begin{aligned} \dot{y}_x &= \dot{y}_a \frac{x}{a} + \Sigma \frac{r\pi c}{a} P_r \sin \frac{r\pi x}{a}, \\ \dot{y}_a &= V_0 \left/ \left(1 + \frac{4T_0 \rho}{\mu M} \right) \right|^{\frac{1}{2}} = V_1. \end{aligned}$$

From the above we find

$$\int_0^a \dot{y}_x \sin \frac{r\pi x}{a} dx = \int_0^a V_1 \frac{x}{a} \sin \frac{r\pi x}{a} dx + P_r \frac{r\pi c}{2}.$$

Since at the beginning of the second state of affairs \dot{y}_x is everywhere zero except at $x=a$, we get

$$P_r = \frac{2a(-1)^r V_1}{c\pi^2 r^2}, \quad . \quad . \quad . \quad . \quad (16)$$

$$\begin{aligned} \dot{y}_x &= V_r \frac{x}{a} - \frac{2V}{\pi} \left\{ \sin \frac{\pi x}{x'} \cos \frac{\pi ct'}{a} - \frac{1}{2} \sin \frac{2\pi x}{a} \cos \frac{2\pi ct'}{a} + \dots \right\}, \\ &\quad . \quad . \quad . \quad (16.1) \end{aligned}$$

where V_r is the velocity of the striking point at time t' .

The above series can be evaluated for particular values of x or t' easily. For instance, when $ct' = \frac{a}{2}$,

$$\begin{aligned} \dot{y}_x &= \frac{x}{a} (V_r - V_1), & 0 < x < \frac{a}{2}, \\ y_x &= \left[\frac{(V_r - V_1)x}{a} + V_1 \right], & \frac{a}{2} < x < a. \end{aligned}$$

Now V_v will differ from V_1 by a very small amount in the time $ct' = \frac{a}{2}$; hence the portion of the string from $x=0$ to $x=\frac{a}{2}$ is at rest, while in the portion from $x=\frac{a}{2}$ to $x=a$ the points will have a velocity slightly less than the initial velocity of the point $x=a$ by an amount proportional to its distance from the fixed end. Similarly the point $x=\frac{a}{2}$ will have a velocity at any time t' ,

$$\dot{y}_{a/2} = \frac{(V_v - V_1)}{2}, \quad 0 < ct' < \frac{a}{2},$$

$$\dot{y}_{a/2} = \frac{(V_v + V_1)}{2}, \quad \frac{a}{2} < ct' < a.$$

Thus we find that the velocity of the point will sometimes be greater than, and sometimes less than, that which would be obtained at the same place when there were no free vibrations. This will repeat after an interval $T = \frac{2a}{c}$.

Solution of (15) :

$$y_a = \frac{V_1 e^{-\frac{k}{2}t'}}{q} \sin qt' - \frac{2V_1 T_0}{\pi c \mu} \sum \frac{1}{r} \sin \frac{r\pi ct'}{a} - \frac{2a^2 V_1 \rho}{c\pi^3 M} \sum \frac{1}{r^3} \sin \frac{r\pi ct'}{a} \dots \quad (17)$$

The last two terms are due to free vibrations.

I.

Calculation of the Pressure due to Impact.

From (13) we obtain, after substituting the values of y_a and \dot{y}_a ,

$$p = V_1 \rho c \left[e^{-\frac{k}{2}t'} \frac{\sin qt'}{q} \left(\frac{c}{a} - \frac{k}{2} \right) + e^{-\frac{k}{2}t'} \cos qt' + \frac{2}{\pi} \left(1 - \frac{T_0}{\mu a} \right) \sum \frac{1}{r} \sin \frac{r\pi ct'}{a} \right]. \quad (18)$$

(18) shows clearly the effect of free vibration, and in the case when μ is infinite (hard hammer),

$$p \text{ (hard hammer)} = V_0 \rho c \left[e^{-\frac{k}{2} t'} \frac{\sin q t'}{q} \left(\frac{c}{a} - \frac{k}{2} \right) + e^{-\frac{k}{2} t'} \cos q t' + \frac{2}{\pi} \sum \frac{1}{r} \sin \frac{r \pi c t}{a} \right]. \quad (19)$$

[Compare Das's formula, Proc. Phys. Lond. xxix. p. 29 (1927).]

The last term in (18) and (19) repeats after an interval of $\frac{2a}{c}$, and the resultant pressure can be obtained by algebraic summation of the two parts. In the case of a slightly elastic hammer we find that the pressure rises by $V_1 \rho c \left(1 - \frac{T_0}{\mu a} \right)$ at intervals of $\frac{2a}{c}$; while in the case of a hard hammer the rise is $V_0 \rho c$. Hence we find that for a slightly elastic hammer the pressure is initially zero; after a short interval t_1 it rises to $2V_1 \rho c$; the subsequent value of the pressure is given by (18). Fig. 1 shows (18) graphically; the sharp rise will diminish with softer hammer, and also with the diminution in the number of the harmonics. If, for instance, only ten harmonics are present, then the sharp peaks will be rounded off and the pressure at $T = \frac{2a}{c}$ will be given by the first term only, the second term being zero.

Amplitude of Free Vibration.—The free vibration is given by

$$y_x = \frac{2aV_1}{c\pi^2} \sum \frac{1}{r^2} \sin \frac{r\pi x}{a} \sin \frac{r\pi c t'}{a},$$

which is a two-step zigzag curve with respect to time at a given point; further, the amplitude is proportional to V_1 , diminishing with μ ; thus, the softer the hammer the smaller is the free vibration. It is also proportional to the vibrating length.

This cannot be true beyond a certain limit, when our assumption $\tan ka = ka$ does not hold good.

II.

Highly Elastic Hammer.

In this case also we assume

$$y_1 = y_a \frac{x}{a} + \sum P_r \sin \frac{r\pi x}{a} \sin \frac{r\pi c t}{a}; \quad \dots \quad (20)$$

but in the equation of motion the term in \ddot{y}_a is retained, but it is treated as small. Hence we find the solution

$$y_a = Ae^{-ut} + Be^{-\frac{k}{2}t} \sin qt + Ce^{-\frac{k}{2}t} \cos qt + \text{small terms,}$$

where k and q have the same values as before and

$$u = -\frac{C\mu}{T_0} \left(1 + \frac{T_0}{\mu a}\right).$$

A, B, and C are constants to be determined from the initial conditions.

$$\text{At } t=0, \quad y_a = \dot{y}_a = 0, \quad p = 0$$

everywhere, and the initial velocity of the hammer is V_0 .

On substituting these conditions in (20) we get

$$A = -C, \quad B = -\frac{A}{q} \left(u + \frac{k}{2}\right).$$

Further,

$$\dot{\xi} = y + \xi - \frac{1}{\mu} \frac{dp}{dt} = V_0 \quad \text{at } t=0.$$

Thus we get

$$A \left\{ u^2 + q^2 + 3\frac{k^2}{4} + ku \right\} + \Sigma P_r \frac{r^2 \pi^2 c}{a^2} (-1)^r = \frac{V_0 \mu c}{T_0}.$$

This is an indeterminate equation, but the value of P_r can be determined from the following considerations:—

- (1) P must be proportional to V_0 .
- (2) It must have the dimension of length.
- (3) The series must be convergent.
- (4) The increase in the value of P_r should diminish A .
- (5) As regards the effect of $\frac{T_0}{\mu}$ on the amplitude of vibra-

tion no definite experimental result is available at present; yet, borrowing the results for slightly elastic hammer, we find that P_r must diminish with softness of the hammer, and also the free vibration should be very small when the hammer mass is small. The effect of softness of the hammer on the free periods of the short piece of wire can be deduced from the results of § 135 *. It is found that when $M=0$,

* § 135, 'Theory of Sound,' i. (Lord Rayleigh).

and μ is finite and great, the free periods are given by

$$\tan ka = -k \frac{T_0}{\mu}$$

When $\mu = 0$ and M is finite, then

$$k \tan ka = \frac{T_0}{Ma^2}.$$

It is found that the effect will depend upon the mass of the hammer and its elasticity. In the case when the mass is small, and at the same μ is large, the natural frequencies are lowered and the harmonic scale is not disturbed; while in the case when M is large, and μ is small, the periods are lowered and at the same time the harmonic scale is disturbed.

From these considerations we find

$$P_r = \frac{V_0 a (-1)^r g}{\pi^3 c r^3}, \quad . \quad . \quad . \quad . \quad (21)$$

$$g = \left(1 + \frac{T}{\mu} \frac{\rho}{M}\right)^{-\frac{1}{2}}.$$

Of course, in the case of a hard hammer, instead of $\frac{1}{r^3}$ we shall have $\frac{1}{r^2}$.

From the above value of P_r ,

$$A(u^2 + q^2) = \frac{V_0 c \mu}{T_0} \left(1 - \frac{g}{\mu a} \sum \frac{1}{r}\right).$$

Now $\sum \frac{1}{r}$ is a divergent series, but when only a few harmonics are present we can find a lower limit of $\sum \frac{1}{r}$, and for ten components the values come out to be 2.92, and (24) becomes

$$A(u^2 + q^2) = \frac{V_0 c \mu}{T_0} \left(1 - \frac{1.86}{\mu a}\right).$$

With the same values for the constants, the term in brackets has a value .88.

The above value of P_r satisfies all the conditions of the problem at $t=0$ except one, viz. :

At $t=0$,

$$\dot{y}_x = \dot{y}_a \frac{x}{a} + \sum \frac{g V_0 (-1)^r}{\pi^2 r^2} \sin \frac{r \pi x}{a},$$

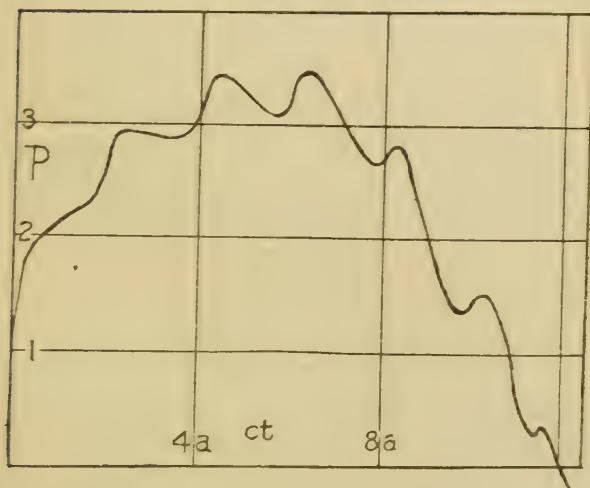
$$\dot{y}_x = 0,$$

but the second term does not vanish as it ought to do, since \dot{y}_x is everywhere zero. This is due to the fact that we have assumed two contradictory things, viz., that there are standing waves in the string which start in the string with zero displacement and zero velocity, which is not possible. Thus during a very short interval of time in the initial stages of motion our formulæ will not represent facts correctly.

The pressure is given by

$$p = T_0 \left[\frac{v_a}{a} + \frac{\dot{y}_a}{c} + \frac{gV_0}{c\pi^2} \sum \frac{1}{r^2} \sin \frac{r\pi ct}{a} \right] \quad (22)$$

Fig. 2.



The graph of (22) with the same values of the constants ($\frac{T_0}{\mu} = 1$) is represented in fig. 2, where only ten terms have been included in the series. This curve shows that the pressure starts from zero value, rises considerably, then follows an irregular course, after which we find a regular rise and fall after an interval $\frac{2a}{c}$.

By comparison with (18) we find that the amplitude of vibration of the short portion diminishes when the hammer becomes softer, just as the rise and fall of pressure become rounded. This effect improves the tone-quality of the lower

notes in the instrument, where the hammer is generally very soft.

Summary and Conclusion.

In the present paper free vibrations of the short portion of the string have been taken into consideration for calculating the pressure due to impact. It is found that in the case of (1) slightly elastic hammer, it rises suddenly at intervals of $\frac{2a}{c}$ by an amount

$$V_0 \rho c \left(1 - \frac{T_0}{\mu a} \right) / \left(1 + \frac{4T_0 \rho}{\mu M} \right)^{\frac{1}{2}};$$

while in the case of a very soft hammer the rise is not sudden and the pressure is given by

$$p = T_0 \left\{ \frac{y_a}{a} + \frac{\dot{y}_a}{c} + \frac{g V_0}{c \pi^2} \sum \frac{1}{r^2} \sin \frac{r \pi c t}{a} \right\},$$

the softness has imposed greater convergence of the terms in series. In a subsequent paper the author intends to indicate the results of experiments carried out to verify the above theory in the case of a highly elastic hammer.

TABLE.

Frequencies of free vibration of the short portion.

$$T_\mu = 12.5, \quad l = 137 \text{ cm.}, \quad P = 0.19, \quad V_0 = 1.04 \times 10^2 \text{ cm./sec.},$$

No.	ϕ :	a cm.	M.	Freq. obs.	Freq. cal.
1.....	1.5×10^{-2}	43	5 grms.	270	274
2.....	1.8 „	43	10 „	276	274
3.....	2.6 „	43	20.5 „	264	274
4.....	2.4 „	72	20.5 „	165	163
5.....	1.6 „	96	20.5 „	125	122

ϕ = Duration of contact.

Physics Department,
Allahabad University,
February 9, 1929.

CVI. *On the Relation of Electronic Waves to Light Quanta and to Planck's Law.* By Sir J. J. THOMSON, O.M., F.R.S.*

SUMMARY.

THIS paper is an attempt to give, by the aid of electronic waves, a physical interpretation of light quanta, their relation to electrons, and the mechanism of light.

WE may regard an electron as consisting of two parts—one, which we shall call the core, carrying a charge of negative electricity equal to e , while the other is a system of waves surrounding the core, the velocity of the core being the group velocity of the waves in its neighbourhood. The wave system as well as the core may have both energy and momentum, and if the core is suddenly stopped the waves will go on and travel away from the charge. A somewhat similar effect was supposed to occur on the original theory of the production of Röntgen rays by the impact of cathode rays; part of the electric and magnetic field surrounding the moving electron in the cathode ray was supposed to travel on after the electron was stopped by its impact with a solid and constitute the Röntgen ray. Thus it would seem that the two parts which at any time make up an electron may, under certain conditions, be separated, and the original system of waves travel far from the core and form a wave system without any electric charge. The view taken in this paper is that light quanta consist of a special type of such wave systems, and thus that the structure of light quanta is very closely connected with that of electrons.

The system of waves associated with an electron which travels like those inside the atom round a closed path is of especial importance in this connexion. Take the case of electrons inside an atom under the action of a charge of positive electricity at its centre. The electronic waves in this case (see *Phil. Mag.* ser. 7, viii, p. 1073) travel round and round the atom, their wave fronts intersecting the paths of the core of the electron at right angles. The waves circulate in closed paths, and as the energy also travels in closed circuits none of it will escape, so that there is no loss of energy by radiation, and the waves when once started will persist for ever.

* Communicated by the Author.

The mechanical properties of this system of waves have many analogies with those possessed by a fluid in which there is vortex motion, the group velocity of the waves corresponding to the velocity of the fluid. For if u, v, w are the components of the group velocity of the electronic waves at any point, then (see *Phil. Mag. loc. cit.*)

$$\oint (u dx + v dy + w dz) = \frac{c^2}{p} \times 2\pi,$$

where the integral is taken round a closed circuit. c is the velocity of light, and p is the frequency of the waves; thus, the integral on the left-hand side is constant both with regard to space and time.

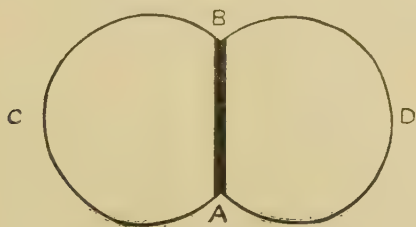
If u, v, w are the components of the velocity of a fluid, the above integral is called the "circulation" round the circuit; it also is constant and depends only on the strength of the vortices enclosed by the circuit. The results which follow from the constancy of the circulation in hydrodynamics have their analogies in the properties of the wave system; thus, for example, it follows from this principle that the motion of the liquid in which the vorticity originated carries along with it the vorticity, hence the motion of the medium in which the waves are produced will carry along with it the system of waves which were started in that medium.

For the sake of illustrating the use of this analogy we will consider one or two special cases. Take first the case of a long straight cylindrical vortex, which may be a cylinder of fluid in which the motion is rotational or a hollow cylinder round which circulation is established. The particles of fluid outside the vortex or hollow describe circles, with their centres on its axis and their planes at right angles to it; the magnitude of the velocity varies inversely as the distance from the axis. The system of waves corresponding to this case would be one where the group velocity of the waves at any point was equal in magnitude and direction to the velocity in the fluid. The fronts of the waves are everywhere at right angles to the group velocity, hence the wave fronts will be radial planes passing through the axis. The distance between two adjacent wave fronts measured along their normal is thus proportional to r , the distance from the axis; but this distance is also proportional to λ , where λ is the wave-length at the distance r ; hence $\lambda \propto r$, but $r \propto 1/u$, where u is the group velocity; hence λu is constant, which agrees with the properties of

electronic waves. The waves, being electrical waves, have momentum, and the momentum at any point is perpendicular to the wave front through that point; hence the momentum will be tangential, and there will be a finite moment of momentum about the axis; the system of waves will, in this respect, resemble a top spinning about its axis. Since the energy is flowing round closed paths there will be no escape of energy.

The system of waves associated with an electron describing an orbit inside an atom is somewhat more complicated than the two-dimensional one we have just described. The hydrodynamical analogue to this system is shown in the figure, which represents the distribution of vorticity in a section of the fluid by a plane through the axis of symmetry AB.

The vorticity consists of closed vortex rings having a portion of their lengths AB in common. The toroidal



surface of which ACBDA is a section is one where the vortex lines are tangential to the surface, and when such a surface moves it carries its vorticity with it (Lamb's 'Hydrodynamics,' p. 186, 5th edit.). Inside this surface the liquid is revolving round the axis AB and has a finite moment of momentum about the axis, and neither energy nor fluid bursts through the surface. All these properties are possessed by the waves associated with an electron moving inside an atom under the influence of a central force varying inversely as the square of the distance, and the electron inside the atom will consist of its core and this system of waves, the core travelling with the group velocity of the waves in its neighbourhood. If the core is detached the wave system will persist and may escape from the atom, carrying with it the energy of the waves and the atmosphere of the electron. I regard this system of waves as a quantum of light, so that, on this view, light quanta are disembodied electrons. After the light quantum has escaped from the atom it will be surrounded by the normal ether and will

travel with the group velocity of waves through this medium, *i. e.*, with the velocity of light. Its direction of motion will be the direction of the group velocity in the ether, which will be determined by the light waves surrounding the quantum; thus these waves will guide the quantum along the path it has to travel. The light quantum on this view consists of a system of electronic waves flowing in closed circuits through a region filled with the atmosphere of the electron. Since the energy moves round the light quantum in closed circuits there is no escape of energy as it moves through space, and, as in the analogous case of vortex motion, the system will carry its medium along with it. No energy is given up by the quantum to the waves of light when once these are established, as they form a system of stationary waves with respect to the quantum. We shall suppose that the light quantum is symmetrical about an axis, and that the circuits round which the waves travel are circles with their centres on this axis and planes at right angles to it.

Any singularity in the wave field inside the light quantum will travel round the quantum with the group velocity of the waves in its neighbourhood. Before the light quantum was detached from the atom the singularity travelling round was the core of the electron. The path of the core will have a definite position in the system of waves, *i. e.*, it will be at a definite distance from the centre of the atom. Since the density of the energy in the waves will be greatest at the envelope of the rays in the wave system, and this is a circle whose radius depends on the energy content of the system, we should expect that this circle would be the path of the core, and also the path of the singularity left when the core is detached.

The motion of such a singularity round the light quantum will produce a periodic effect whose period is the time taken by it to travel round the circuit, and is thus equal to the length of the circuit divided by the group velocity of the electronic waves. We regard this period as that of the waves of light associated with the quantum. There is also another period associated with this quantum—that of the electronic waves circulating round it. This period is infinitesimal in comparison with that of the waves of light; we shall see, however, that these electronic waves are of fundamental importance in determining the relation between the period of the light waves and their energy.

To determine this relation we proceed as follows:—the time taken to describe its circuit by a singularity at a distance r from the axis of the quantum is $2\pi r/u$ if u is the group

velocity of the waves. Hence the frequency of the disturbance, i. e., ν the frequency of the light waves, is given by

$$\nu = \frac{u}{2\pi r} = \frac{u^2}{2\pi ur}.$$

If V is the phase velocity of the *electronic waves inside the light quantum*, L their wave-length, and p their frequency,

$$u = \frac{c^2}{V} = \frac{c^2}{pL};$$

so that

$$2\pi r u = \frac{c^2}{p} \frac{2\pi r}{L}.$$

Now the circuit $2\pi r$ must contain an integral number of wave-lengths; hence

$$\frac{2\pi r}{L} = n \text{ (an integer) ;}$$

so that

$$\nu = \frac{u^2}{2\pi r u} = \frac{pu^2}{nc^2}. \quad . \quad . \quad . \quad . \quad (1)$$

We now proceed to calculate E , the energy in the system of waves inside the light quantum. The energy in the electronic system in the atom before the core was detached was equal to the energy of the core + E . Suppose that this electronic system arose from the falling of an electron from a great distance into the atom. If m is the mass of the electron, u its velocity, then when the electronic system is inside the atom and the core is describing a circle of radius r

$$mu^2 = \frac{e^2}{r}.$$

If the core had retained all the kinetic energy it acquired in its fall, then, neglecting relativity corrections,

$$\frac{1}{2}mu^2 = \frac{e^2}{r};$$

thus the core has lost by its fall an amount of energy equal to $mu^2/2$. We suppose that this has gone into the wave system, so

$$E = \frac{1}{2}mu^2 + E_0,$$

where E_0 is the energy in the atmosphere of the electron when it is at rest and at an infinite distance from the atom; E_0 is a constant, and will not appear in equations representing the transference of energy between light quanta or between a light quantum and an electron.

Thus, when the electronic system is in a steady state, the energy in the waves is equal to the kinetic energy of the core, and is thus equal to $mu^2/2$, where m is the mass of an electron and u the group velocity of the waves at the core of the electron.

From (1)

$$E = \frac{1}{2} \frac{mnc^2}{p} \nu + E_0.$$

If m_0 is the mass of an electron at rest, and p_0 the frequency of its vibration when in that state,

$$\frac{m}{p} = \frac{m_0}{p_0};$$

hence

$$E = \frac{n}{2} \frac{m_0 c^2}{p_0} \nu + E_0,$$

which is equivalent to

$$E = \frac{n}{2} \frac{m_0 c^2}{p_0} \nu = \frac{n}{2} h \nu, \quad \text{if} \quad h = m_0 c^2 / p_0.$$

The resemblance of this equation to Planck's Law is obvious, in fact it becomes identical with it when $n=2$ *. On the view we have taken Planck's relation is a necessary consequence of the existence of electronic waves.

We can estimate the size of the light quanta as follows :—

Taking the case $n=2$, we have by (1)

$$\nu = \frac{pu^2}{2c^2} \quad \text{and} \quad \nu r^2 = \frac{pu^2 r^2}{2c^2} = \frac{c^2}{2\pi^2 p}.$$

When u/c is small, as it is for visible light, p is nearly equal to p_0 , so that this equation becomes

$$\nu r^2 = \frac{h}{2\pi^2 m_0}.$$

* If the time of vibration of the light quantum were measured by the time a disturbance took to travel over a distance equal to the wave-length of the electronic waves inside the quantum, n would not appear.

Now $\nu=c/\lambda$, where λ is the length of the light waves, so that

$$r^2 = \frac{h}{2\pi^2 cm_0} \cdot \lambda,$$

$$r^2 = \frac{10^{-10}}{8} \lambda,$$

$$r = .35\sqrt{\lambda} \times 10^{-5}.$$

Thus for the hydrogen lines

$$\text{H}_\alpha \quad r = 2.8 \times 10^{-8} \text{ cm.},$$

$$\text{H}_\delta \quad r = 2.24 \times 10^{-8} \text{ cm.}$$

Thus the quanta for visible light are of atomic dimensions. Since $r \propto \sqrt{\lambda}$, the size of the quantum varies much less rapidly than the wave-length.

If the light quanta have a finite size there will be a limit to the rate at which energy can be transmitted by light waves. For if r is the linear dimension of the quantum, the maximum number of quanta per unit volume may be taken as $1/r^3$. The energy in each quantum is hc/λ , so that the maximum energy per unit volume is $hc/\lambda r^3$, and the maximum rate at which energy can be transmitted across unit area is $hc^2/\lambda r^3$, or, substituting the value of r ,

$$1.3 \times 10^{11} \cdot \frac{1}{\lambda^{\frac{5}{2}}} \text{ ergs per sec. per square cm.}$$

Since the energy in the light quantum is equal to

$$\frac{m_0 c^2 \nu}{p_0},$$

its mass will equal $m_0 \nu/p_0$.

Moment of Momentum of a Light Quantum about its Axes.

The momentum per unit volume in the field of the electronic waves in the quantum is (see '*Beyond the Electron*,' p. 41) equal to

$$(\text{energy per unit volume in the wave-field})/V,$$

where V is the phase velocity of the electronic waves. The moment of momentum about the axis is equal to

$$\frac{r}{V} (\text{energy per unit volume}),$$

but $2\pi r = nL$, where L is the length of the electronic wave, and therefore $r/V = n/2\pi \cdot p$. Thus the total moment of momentum is equal to

$$\frac{n}{2\pi p} (\text{energy of quantum}) = \frac{n^2 h}{4\pi} \frac{v}{p}.$$

Since the light quantum has a finite moment of momentum about its axis, to fix the quantum we require, in addition to its energy, a vector to fix the position of its axis.

The case we have considered is one where one complete electronic system, *i. e.* core and waves, is deprived of its core and the whole of its wave-system goes to form the light quantum. The genesis of light quanta from atoms is somewhat more complicated than this. By doing work on an electronic system at one energy-level it is transferred to a higher level, and when it returns to its original level the work done to transfer it to the higher level appears in a light quantum. Whilst the electronic system is passing from the lower to the higher level it is not in a steady state, and there is no coordination between the core and the waves; the velocity of the core is not necessarily equal to the group velocity of the waves. We may suppose that during the transference the energy in the wave-system does not alter

and is equal to $\frac{1}{2} \frac{e^2}{r_1}$ where r_1 is the radius of the first

energy-level. We have seen that when there is equilibrium between the wave-system and the core the energy in the waves is equal to $e^2/2R$, where R is the distance of the core from the centre. In the steady state at the second energy-

level r_2 the energy in the waves round the core is $\frac{1}{2} \frac{e^2}{r_2}$; hence

in this position there is an excess of energy in the wave system coming from the first energy level equal to

$\frac{1}{2} e^2 \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$. Let us suppose that this excess splits off

as a separate system, which is not so closely held by the core as the other. When the electron falls back to its former

level it carries with it the waves whose energy is $\frac{1}{2} \frac{e^2}{r_2}$, and the system whose energy is $\frac{1}{2} e^2 \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$ is free to escape as

a light quantum. The fall of the core from r_2 to r_1 leaves free an amount of energy $\frac{1}{2} \left(\frac{e^2}{r_1} - \frac{e^2}{r_2} \right)$ which goes into the

system of waves attached to the core and increases its energy from $\frac{1}{2} \frac{e^2}{r_2}$ to $\frac{1}{2} \frac{e^2}{r_1}$, the value it has when core and waves are in equilibrium.

The quantum of light on this theory is a system of electronic waves flowing round a closed circuit. This circuit must have been closed before the waves were detached from the core; this is the case when the electron which supplied the waves was one of the electrons in an atom. It would not be so with a free electron like a cathode ray. When the core is detached from this system the only definite frequency in the wave-system is the frequency of the electronic waves themselves, and the Röntgen rays produced by the impact of cathode rays against a target must owe any definite frequency they possess to the influence of the target itself. The path of the electron, and therefore of the electronic waves, through the target, owing to collisions with the molecules of the target, will be very irregular, and may contain a loop; the length of this loop must be an integral multiple of the wave-length of the electronic waves. The loop may be detached and escape, carrying with it the electronic waves which circulate round it. This system is of the same character as that given out by an atom when emitting one of its spectral lines. It will be quantized, because quantization is the result of having an integral number of wave-lengths round the circuit, and Planck's relation between period and energy will hold. As the velocity of the electron in the target is continually changing through collisions, the energy in the light quanta it emits will not be fixed, but will vary over a very wide range, and thus constitute a continuous spectrum. This corresponds to the white spectrum of the Röntgen rays. Those constituents of this spectrum, which are of any particular wave-length, will be of exactly the same character as the constituents of a monochromatic spectrum of that wave-length.

The same thing applies to the continuous spectrum given out by a hot body; this is due to the irregular motions of electrons in the body. We can also put the same reasoning into a form which brings it into close connexion with a method which has long been used to describe the origin of electrical waves. Consider an electron and a positive charge. Tubes of electric force stretch from the electron to the charge, and in these tubes lies the energy of the field. For any of this energy to travel away into space without carrying the electron

with its tubes must break away, and to do this they must form a loop. When the electron moves about the tubes move with it and may get looped. Since electronic waves are running along the tubes, the length of the loop must be an integral multiple of the length of these waves. When the loop gets detached electronic waves are travelling around its closed circuit, and we have just what on the theory we are considering is a light quantum. The frequency of the light is proportional to the energy in the quantum, and, as this depends on the velocity of the electron, the spectrum given by an irregularly moving collection of electrons will be continuous.

CVII. Notices respecting New Books.

Mineralogy. An Introduction to the Scientific Study of Minerals.

By Sir HENRY A. MIERS, M.A., D.Sc., F.R.S. Second Edition, revised by H. L. BOWMAN, M.A., D.Sc. [Pp. xx+658, with 761 illustrations.] (London: Macmillan and Co. 1929. Price 30s. net.)

THE second edition of Sir Henry Miers's 'Mineralogy' has been revised by Dr. Bowman, his successor in the Waynflete Chair of Mineralogy in Oxford. It follows the original plan of the book, with such changes and additions as have seemed necessary after the lapse of twenty-seven years. The volume is restricted to an account of the general properties of minerals,—crystalline, physical, and chemical—to the description of the more important mineral species, and to an account of the methods for the determination of minerals. No systematic account of the occurrence of minerals, their geological distribution and origin are given.

The volume is so well known to students of mineralogy that no detailed review of the contents of the second edition is necessary, the more so as the changes from the first edition are not extensive. The most important addition is an account of the analysis of crystal structure by means of X-rays. This is limited to twenty pages and might well have been expanded somewhat in view of the importance and power of the method. The idea of the wave-surface has been introduced into the chapter on the optical properties of crystals; this is a valuable and important addition. Numerous minor changes have been made and many new figures have been added.

The tables at the end of the volume—comprising a list of the principal minerals, tables of reactions, and tables of physical properties are very convenient for reference purposes. The index is excellent.

This text-book is one which no student of metallurgy can afford to dispense with.

The Earth: its Origin, History, and Physical Constitution. By HAROLD JEFFREYS, M.A., D.Sc., F.R.S. Second Edition. [Pp. x+346, with 16 figures.] (Cambridge: at the University Press. 1929. Price 20s. net.)

THE developments in the subjects dealt with in this volume during the four years which have elapsed since the first edition appeared have necessitated an increase in size and price of the new edition of about twenty-five per cent. The portion devoted to seismology has been completely rewritten and considerably extended. A full account is given of the important researches of Gutenberg and others. A chapter is devoted to general considerations and the structure of the upper layers, and another to the information as to the Earth's interior that is provided by seismology. The subject is thus brought into a more intimate relation with the other subjects dealt with. In particular, the conclusions provided by seismology have an important bearing on the thermal history of the Earth, and the chapter devoted to this subject has been rewritten.

The general question of isostasy receives a more extended discussion, and more definite conclusions are drawn as to the nature and degree of isostatic compensation. More attention has been given to the bodily tide in the Earth and to bodily tidal friction. Much fuller discussion is given of the contraction theory of mountain formation, and of theories of the origin of the continents and of the permanence of the continents. In addition, numerous minor additions and revisions have been made.

The result of these additions has been to increase considerably the value of the book. There is no other book which surveys in so authoritative a manner the work in many branches of geophysics and coordinates them. Astronomy, geology, seismology, and the general properties of matter are all called upon to contribute evidence. The style is lucid and clear, the mathematical portions are not unduly heavy, and the volume is eminently readable.

A new appendix entitled "The Relation of Mathematical Physics to Geology" has been added, replacing that on the hypothesis of the infinite deformability of the Earth by small stresses. It contains much common sense, and can be recommended to the serious attention of those geologists who are inclined to undervalue the evidence which can be derived from mathematical considerations.

Magnetism. By EDMUND C. STONER, Ph.D. (Methuen's Monographs on Physical Subjects.) [Pp. vii—117, with 20 diagrams.] (London: Methuen and Co. 1930. Price 2s. 6d. net.)

THEORIES of magnetism have been profoundly affected by modern views of atomic structure. This small monograph starts appropriately, therefore, with a consideration of the magnetic properties of atoms from the point of view of quantum mechanics. Diamagnetism, paramagnetism, and ferromagnetism are then dealt with in succession. A final chapter deals with the magnetic properties of the elements.

It will be seen that the book is intended for those who have some previous knowledge of the subject and describes some of the modern investigations and the present outlook on the subject. The development is necessarily somewhat condensed, though lucid and logical. The *résumé* of the modern outlook on the subject provides a fitting introduction to the subject for those who wish to pursue it further, and will be appreciated by workers in kindred subjects who find it difficult to keep in close contact with modern rapid developments. A summary of the more important books and articles on the subject of magnetism is given, and there are footnote references to more recent work, of which an account is not available in the books quoted.

Catalogue of Lewis's Medical and Scientific Circulating Library. Part I. Authors and Titles. Part II. Classified Index of Subjects, with Names of Authors who have written upon them. New edition, revised to the end of 1927. [Pp. 408+166.] (London: H. K. Lewis and Co. 1928. Price 15s. net; to subscribers, 7s. 6d. net.)

MESSRS. LEWIS'S Circulating Library was established in 1848, primarily to supply the needs of members of the medical profession. It has developed into a general scientific library, in which may be found books of scientific or philosophical interest which are not to be found in the ordinary circulating library. The catalogue deserves to be better known and has distinct value for reference purposes, quite apart from the library itself.

The first part is arranged alphabetically according to authors and gives the full name of the work, the editor, size of page, price, and date of publication. The second part consists of a useful classified subject catalogue; under each subject the arrangement is according to authors, with the addition of a word or two to indicate the name and scope of the book, and the date of publication. The latter enables most recent books to be identified at a glance. For such a classified index considerable cross referencing is desirable: this has been incompletely done. Under astronomy, for example, the names of neither Jeans nor Eddington are to be found. Eddington's 'Internal Constitution

of the Stars' is indexed under Stars, not under Astronomy. Arrhenius's 'Destinies of the Stars,' on the other hand, is given both under Astronomy and Stars. Other examples could be multiplied. In view of the incompleteness of the cross-referencing, a summary list of the headings in Part II. would save much turning of pages.

The library does not contain any foreign books, unless translated. Old standard classical works are also, in general, missing. These omissions limit the usefulness of both library and catalogue. Subject to these limitations, the Catalogue appears to be reasonably complete. Messrs. Lewis are to be congratulated upon the production of a useful volume.

CVIII. *Proceedings of Learned Societies.*

GEOLOGICAL SOCIETY.

[Continued from p. 1064.]

March 12th, 1930.--Prof. E. J. Garwood, M.A., Sc.D., F.R.S.,
President, in the Chair.

THE following communications were read:—

1. 'The Age of the Midland Basalts. By Roy Woodhouse Pocock, M.Sc., F.G.S.

Basic igneous rocks of Lower Carboniferous age are known to occur in the Bristol district, at Little Wenlock in Shropshire, in North Staffordshire, Derbyshire, and Cumberland. Rocks of similar type are present in the Upper Carboniferous of the Midlands, but hitherto no definite evidence of their age has been produced.

The object of this paper is to bring together all available data bearing on the age of the various igneous masses in the Upper Carboniferous, such as those of Kinlet, Shatterford, Clee Hill, Claverley, Rowley Regis, Wednesfield, etc.

The Kinlet basalt is shown, by the evidence of certain exposures, to be a flow of Yorkian age, probably near the top of that division, and this conclusion is supported by evidence from a recent boring east of the main basalt-mass.

The Shatterford basalt is shown to be extrusive, on the evidence of its relationship to the overlying and underlying Coal Measures.

The horizon of this basalt is somewhat lower in the Yorkian than that of Kinlet.

At Clee Hill the evidence shows the basalt to be extrusive, and that it suffered erosion during Yorko-Staffordian time.

The basic igneous rock in the Claverley boring is mentioned, as showing the association of igneous activity with the Coal Measures in the tract between Wyre Forest and South Staffordshire.

The Rowley basalt appears to lie just below the position of the unconformity, if such is present in the district, between the Etruria Marl and the Halesowen Beds. The evidence available for deciding whether the Rowley mass is extrusive or intrusive is discussed.

The rocks of Wednesfield and Pouk Hill are known to be definitely intrusive.

The age of these South Staffordshire rocks is considered, their close connexion (petrologically, stratigraphically, and structurally) with the Shropshire basic rocks is emphasized, and the conclusion is reached that the Shropshire and South Staffordshire basalts are of the same general age: namely, Yorko-Staffordian, whether extrusive or intrusive.

A volcanic belt is known to traverse the Midlands from Hanter and Stanner in Radnorshire to the Ashby Coalfield area. The main movement along this zone took place in Yorko-Staffordian time, and the basaltic masses in question are, without exception, situated along it.

2. 'The Origin of the Etruria Marl.' By Thomas Robertson, B.Sc., Ph.D.

The Etruria Marl Group of the Upper Coal Measures in the Midlands is mainly composed of chocolate-coloured to purple clay, mottled with green, yellow, etc., alternating with greenish sandstones (Espley Rock).

There is no earlier sedimentary rock in the district that could by decomposition yield so characteristic a facies as this Group presents, nor does the Group show characters definitely known to belong to deposits derived from arid regions.

In appearance and composition the Etruria Marl strongly resembles the denudation-products of basalt and basic tuff, and further examination shows that it contains fragments of basalt; that its position in the geological sequence is the same as that of the Coal-Measure vulcanicity in the Midland Province; and that it is best developed in those portions of the basins of deposition towards which the denudation-products of the Midland basalts would flow.

It is thus concluded that the Etruria Marl was formed by the decomposition of the Midland basalts; and certain subsidiary deductions are made regarding the general geological significance of the Etruria Marl facies.

March 26th, 1930.—Prof. E. J. Garwood, M.A., Sc.D., F.R.S.,
President, in the Chair.

The following communication was read:—

‘A Classification of some Rhyolites, Trachytes, and Phonolites from part of Kenya Colony, with a Note on some associated Basaltic Rocks.’ By Walter Campbell Smith, M.C., T.D., M.A., Sec.G.S.

Comparison of specimens collected on two expeditions by Prof. J. W. Gregory in 1893 and 1919, previously described by Dr. G. T. Prior (1903) and Miss A. T. Neilson (1921), supported by some new analyses, has led to a revision of the somewhat confused nomenclature. The rocks are classified as follows:—

Rhyolites	{ Comendites. Pantellerites.
Trachytes	{ Soda-trachyte (Gibélé type) Washington. Pantelleritic trachytes and kataphorite-trachytes.
Phonolites	{ (Kenya type) Prior and Neilson. (Losuguta type) Prior. Kenytes.

The pantelleritic trachytes include most of the phonolitic quartz-trachytes of Prior.

The ‘Kapitian phonolites’ are shown to be identical with the kenytes of Mount Kenya, which are found to contain phenocrysts of nepheline, as well as the more conspicuous well-known anorthoclase. Prior’s contention that the kenytes are basic members of the phonolite series is confirmed.

Basaltic rocks associated with the phonolites are relatively scarce. They include mugearites, alkali-basalts, and porphyritic types with abundant phenocrysts of augite and olivine.

April 9th, 1930.—Prof. E. J. Garwood, M.A., Sc.D., F.R.S.,
President, in the Chair.

Prof. LÉON W. COLLET, For. Corresp. G.S., delivered a lecture on the Structure of the Canadian Rockies. The Lecturer said that Prof. K. F. Mather and Prof. P. Raymond of the Geological Department of Harvard University supervised in 1929, with the collaboration of Dr. Parejas of Geneva, a month’s Summer Course in the Canadian Rockies to investigate the stratigraphy along the Athabasca Valley (Jasper National Park) and round Mount Robson (B.C.). This course of 20 students did, in fact, the reconnaissance work for a two months’ expedition (financed by the Shaller Fund) in which the structures were studied under the leadership of the Lecturer, with the collaboration of Dr. Parejas and A. Lombard.

The Lecturer presented and described the section that he had made with Dr. Parejas along the Athabasca Valley, from the eastern border of the Rockies to Yellow Head Pass, that is, a complete section across the Rockies. This mountain-chain is made up of seven 'blocks' thrust one over the other from west to east, and separated by 'clean-cut thrusts' of the type of the North-West Highlands of Scotland.

From east to west, at Boule Roche the Carboniferous is overriding the Cretaceous of the Great Plains. On the eastern side of Roche Ronde one sees the Upper Cambrian on the Cretaceous of the former block. At the foot of Greenock Mountain, the Devonian is thrust over the Triassic. In Vine Creek the Devonian again overrides the Triassic. On the eastern side of Gargoyle the Devonian can be seen resting on the Jurassic. On the eastern side of Chetamon Mountain the Cambrian is pushed over the Carboniferous, and last but not least, at the foot of Pyramid Mountain, the pre-Cambrian is thrust over the Devonian.

In the Mount Robson region the Lecturer and Dr. Parejas detected a thrust-plane on the eastern side of Titkana Peak that had been imagined by Walcott, owing to the tremendous thickness of the Cambrian strata.

Comparing the structure of the Canadian Rockies with the structure of the Alps, the Lecturer showed that we were dealing with two different types of folding. The structure in blocks of the Canadian Rockies corresponds to Argand's 'ground folds' (*plis de fond*), while the Alps are made up of 'recumbent folds' developed in a geosyncline. In the Canadian Rockies the energy necessary for the folding was much greater than in the Alps, for in the former the strata have been cut into blocks as far down as the pre-Cambrian.

The Lecturer showed, moreover, that the Ordovician and the Silurian are missing in the eastern part of the Rockies. The Ordovician alone appears in the western part. He considered that this stratigraphical gap was a repercussion, across the Canadian Shield, of the Caledonian folding of the Canadian Appalachians.

The results arrived at by his expedition confirm the structure of the Canadian Rockies as shown by Prof. R. A. Daly's section along the 49th parallel, and the views expressed by Prof. E. Argand on the geology of North America in several chapters of his well-known '*Tectonique de l'Asie*'.

[*The Editors do not hold themselves responsible for the views expressed by their correspondents.*]

FIG. 1.

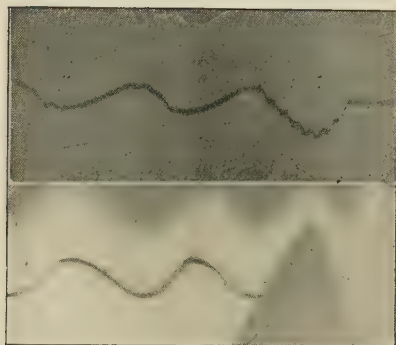
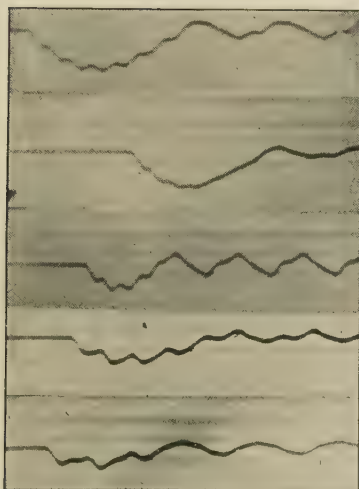


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